STRESS ANALYSIS OF SUPERCONDUCTING MAGNETS

By

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To my beloved Parents

and

dear sister, Niloofar

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Closed form solutions to the stress analysis for an elastic coil of superconducting magnets, which include isotropic Green's functions solution and orthotropic generalized plane strain analysis, are presented. The prediction of stress and strain is essential for both mechanical and electrical design of high field superconducting magnets. The generalized plane strain analysis provides an analytical formulation for the axisymmetric state of stress and strain in the midplane of a superconducting magnet, where shear stress is negligible. This formulation takes into account the effect of the axial body force. The full axisymmetric stress analysis in terms of Green's functions provides the components of stress throughout the coil and includes the shear stress in addition to the normal stresses. Green's functions method permits the development of a solution irrespective of the type of the field or its distribution within the coil. Green's functions are derived by using finite integral transforms in an interval appropriate for a coil, e.g., finite Hankel transform. A comparison of the analytical results with the finite element methods is presented to confirm the validity of the solutions. The results show good agreement for both generalized plane strain analysis and Green's functions solution with the results of finite element methods.

CHAPTER 1
INTRODUCTION

Stress analysis for a high field solenoid magnet is an important aspect of the overall magnet design. The prediction of stress and strain is essential for both mechanical and electrical design of high field solenoid magnets. These magnets are designed in a variety of configurations. A superconducting magnet is one example of such magnets, which can be treated as a combination of several coils, where each coil may be reinforced by a nonconducting layer. Depending upon the geometrical specifications of a coil, magnetic fields may behave differently. These fields result in magnetic body forces, and thus stresses, in all directions. Traditionally only the tangential component of stress has been considered for design and failure analysis. Recent experimental results indicate that in addition to the normal stresses, shear stress also contributes to failure. The value of shear stress has been determined to be small in the midplane and becomes larger toward the ends of the coil. In the present analytical solutions, shear stress is assumed to be negligible and the use of finite element methods is time consuming especially for the case of designing a magnet with many layers. As a result a three-dimensional closed form solution is desired to develop design criteria and understand the distribution of stresses (including shear) through a solenoid magnet. Such a closed form solution can be used in a numerical scheme for multilayered magnet structures.

In this work, the general closed form solutions, using Green's function method and generalized plane strain condition, are derived for high field solenoid magnets. These solutions are applied to the important case of superconducting magnets. The generalized plane strain analysis is performed for the midplane of an elastic orthotropic coil of a solenoid magnet where shear stress is negligible. The full axisymmetric stress analysis for

an elastic isotropic coil is derived using the Green's functions solution. This analysis is not limited to the midplane and can be used for any type of solenoid magnet.

Historically, magnet stress analysis has been presented as a plane strain or plane stress problem. Here, a better treatment in terms of generalized plane strain is developed, which takes into account the effect of the axial body force. The generalized plane strain provides an accurate prediction of all normal components of stress and as a result is a good approximation to the general three-dimensional solution. For the first time a three-dimensional stress analysis for an isotropic axisymmetric coil with finite intervals in both radial and axial directions and subjected to body forces is derived. The use of Green's functions is the best possible technique for providing solutions for any type of body forces. In this analysis, the boundary conditions for the coil become complicated and cannot be satisfied by the traditional methods. A new approach is used to satisfy the boundary conditions which results in a complementary solution.

The use of Green's functions solution is not limited to magnetic body forces. It can be applied to other axisymmetric elasticity problems with finite intervals. The Green's functions solution can be used for inclusion problem in composite materials where eigenstrains may be considered as body forces.[1-4] The Green's functions solution can also be applied to crack problems, due to residual stresses, in composite materials, where difference between the thermal expansion coefficients of the fiber and the matrix results in fictitious body forces.[5-6]

This dissertation is divided into six chapters and five appendices. In Chapter two superconductivity is briefly explained and magnetic fields for a circular wire are derived. In Chapter three magnetic and thermal stress analyses based upon generalized plane strain assumption for an elastic orthotropic coil are introduced and the arbitrary constants in the magnetic stress formulation are determined for different kinds of boundary conditions. The derivation of the arbitrary constants for the thermal stress formulation is included in Appendix A.

The main theme of this work is in Chapter four, where the three-dimensional axisymmetric stress analysis in terms of Green's functions is performed. The problem is formulated in terms of radial and axial stress functions. The partial differential equations for stress functions are derived from the fundamental field equations (a detailed derivation of the partial differential equations is given in Appendix B). The radial stress function is solved in terms of the radial Green's function by using the finite Hankel transform of order one and the finite Fourier cosine transform. The axial stress function is derived in terms of the axial Green's function by using the finite Hankel transform of order zero and finite Fourier sine transform. The derivation of the finite Hankel transform is also included in this chapter (proof of orthogonality property of the finite Hankel transform is provided in Appendix C). The boundary conditions are satisfied by a complementary solution for the axial stress function. By using a new approach, the complementary solution is derived for traction free boundary conditions. The derivation of the complementary solution is not limited to the traction free boundary conditions and can be determined for any kinds of boundary conditions. The final solution for stresses is obtained from the radial Green's function together with the superposition of the axial Green's function and the complementary solution. Moreover in this chapter, for each of the derivations of similar equations only one derivation of an equation is given and the rest are provided in appendices (Appendix D, Appendix E and Appendix F).

In Chapter five the results of stress analysis for the example magnets are presented. These results are compared to the finite element methods to confirm the validity of the analytical stress analysis. In Chapter six the conclusion is provided and some future work is proposed.

CHAPTER 2
SUPERCONDUCTING MAGNETS

Introduction

A superconducting magnet is a nested set of individual cylindrical superconducting coils on separate support structures (Fig. 2.1). A superconducting coil may have additional support structure (reinforcement) on the outer diameter to constrain the coil. There is a

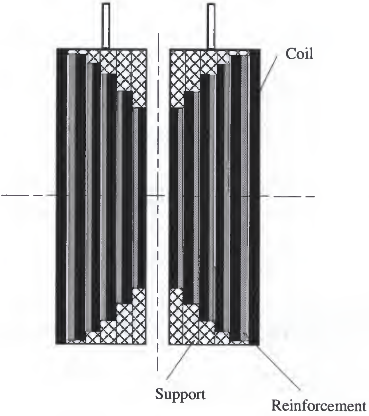

Figure 2.1 Schematic diagram of a representative superconducting magnet.

long history of attempts to understand and improve the performance of superconducting magnets. The tendency of coils to quench prematurely at relatively low fractions of the

critical current is often attributed to mechanical issues. Stress analysis, and a knowledge of stress, strain and displacement of the windings, is therefore essential to the design of superconducting magnets.

Superconductivity

The complete absence of electrical resistivity for the passage of current below a certain "critical" temperature (less than 21 °K) is the basic premise of superconductivity. In addition to critical temperature, the critical field and critical current density are two parameters that define a critical surface. Fig. 2.2 shows a surface in a three dimensional space whose axes are temperature T, magnetic field B and current density J.

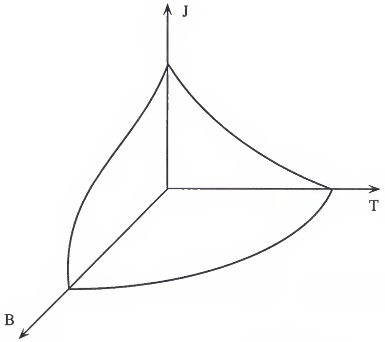

Figure 2.2 Critical surface of a typical superconducting magnet.

The superconducting phase exists only for the points below the surface; points above the surface correspond to the presence of resistivity.[8] The critical temperature, T_c, of a magnet is the temperature where resistivity suddenly drops from a high value (10 μΩ.cm) to a low

value (less than 10^{-9} $\mu\Omega$.cm). For comparison, resistivity of the purest copper is about 10^{-3} $\mu\Omega$.cm so for all practical applications resistance of a superconductor is assumed to be zero. The transition between normal state and superconducting phase takes place at the critical field, B_c.[9] The critical current density, J_c, is the maximum current density that can be applied to a conductor in the superconducting phase. The critical current density of a superconducting wire is strain dependent.[10] This is specially true for Nb_3Sn superconductors as shown in Fig. 2.3. For most superconducting wires, the critical current is observed to increase initially with tensile strain and then decrease with increasing tension.

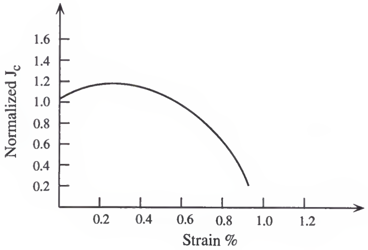

Figure 2.3 Uniaxial strain dependence of the critical current density of a Nb_3Sn superconductor.[8]

Magnetic Fields

The state of stress in a coil subjected to a uniform current density is a direct result of magnetic fields. Magnetic fields for a solenoid coil may be calculated by integrating the contributions from individual circular current filaments. The magnetic field vector, **B**, from a circular current loop is calculated from the vector potential **A**, where $\mathbf{B} = \nabla \times \mathbf{A}$.

The vector potential **A** at point P (see Fig. 2.4) due to a current I has only a tangential component given by Eq. (2.1).[11]

$$A_{\varphi(r,z)} = \frac{\mu_0 I}{4\pi} \oint_c \frac{a \cos\varphi ds}{|\mathbf{R}|} \tag{2.1}$$

where μ_0 is the magnetic permeability, a is the radius of the loop and **R** is a vector from point P to any point on the loop. Vector **R** may be written in the form $\mathbf{R} = \mathbf{oP} - \mathbf{oB}$. In cylindrical coordinates (r, φ, z), **oB** and **oP** are expressed by $a\cos\varphi\mathbf{e}_r + a\sin\varphi\mathbf{e}_\varphi$ and $r\mathbf{e}_r + z\mathbf{e}_z$, respectively. Using these expressions, we may write

$$\mathbf{R} = (r - a\cos\varphi)\mathbf{e}_r - a\sin\varphi\mathbf{e}_\varphi + z\mathbf{e}_z$$

and, hence,

$$|\mathbf{R}| = \sqrt{(r - a\cos\varphi)^2 + a^2\sin^2\varphi + z^2}$$

or by expanding the $(r - a\cos\varphi)^2$,

$$|\mathbf{R}| = \sqrt{r^2 + a^2 + z^2 - 2ar\cos\varphi}. \tag{2.2}$$

The closed path of the integral in Eq. (2.1) is a circle, thus, $ds = ad\varphi$. Introducing Eq. (2.2) into Eq. (2.1) gives

$$A_{\varphi(r,z)} = \frac{\mu_0 I}{4\pi} \int_{-\pi}^{\pi} \frac{a\cos\varphi d\varphi}{\sqrt{r^2 + a^2 + z^2 - 2ar\cos\varphi}},$$

or since the expression inside the integral is an even function in terms of φ

$$A_{\varphi(r,z)} = \frac{\mu_0 I}{2\pi} \int_{0}^{\pi} \frac{a\cos\varphi d\varphi}{\sqrt{r^2 + a^2 + z^2 - 2ar\cos\varphi}}. \tag{2.3}$$

The substitution of $\varphi = \pi - 2\theta$ into Eq. (2.3) yields Eq. (2.4).

$$A_{\varphi(r,z)} = -\frac{\mu_0 I}{\pi} \int_0^{\pi/2} \frac{a\cos 2\theta d\theta}{\sqrt{r^2 + a^2 + z^2 + 2ar\cos 2\theta}} \tag{2.4}$$

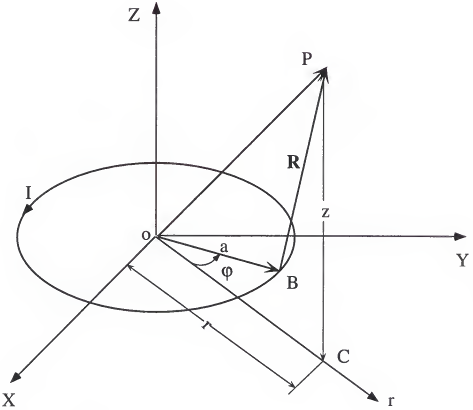

Figure 2.4 Magnetic field at any point in space from an ideal circular loop.

By substituting $\cos 2\theta = 1 - 2\sin^2\theta$ in Eq. (2.4), we may write

$$A_{\varphi(r,z)} = -\frac{\mu_0 I}{\pi} \int_0^{\pi/2} \frac{a(1 - 2\sin^2\theta) d\theta}{\sqrt{(r+a)^2 + z^2 - 4ar\sin^2\theta}}. \tag{2.5}$$

Introducing a new parameter $k = \frac{2\sqrt{ar}}{\sqrt{(r+a)^2 + z^2}}$ into Eq. (2.5), yields

$$A_{\varphi(r,z)} = -\frac{\mu_0 I}{\pi} \frac{1}{\sqrt{(r+a)^2 + z^2}} \int_0^{\pi/2} \frac{a(1 - 2\sin^2\theta) d\theta}{\sqrt{1 - k^2\sin^2\theta}},$$

or

$$A_{\varphi(r,z)} = -\frac{\mu_0 I}{\pi}\frac{a}{\sqrt{(r+a)^2+z^2}}\left\{\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\right. \tag{2.6}$$

$$\left. +\frac{2}{k^2}\left[\int_0^{\pi/2}\frac{\left(1-k^2\sin^2\theta\right)d\theta}{\sqrt{1-k^2\sin^2\theta}} - \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\right]\right\}.$$

The integrals in Eq. (2.6) are in form of elliptic integrals. Thus, we may write

$$A_{\varphi(r,z)} = -\frac{\mu_0 I}{\pi}\frac{a}{\sqrt{(r+a)^2+z^2}}\left\{K_{(k)}+\frac{2}{k^2}\left[E_{(k)}-K_{(k)}\right]\right\} \tag{2.7}$$

where $K_{(k)}$ and $E_{(k)}$ are elliptic integrals of the first and second kind given by Eqs. (2.8) and (2.9), respectively.

$$K_{(k)} = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{2.8}$$

$$E_{(k)} = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta \tag{2.9}$$

The substitution of $\frac{1}{\sqrt{(r+a)^2+z^2}} = \frac{k}{2\sqrt{ar}}$ into Eq. (2.9), yields

$$A_{\varphi(r,z)} = \frac{\mu_0 I}{k\pi}\sqrt{\frac{a}{r}}\left\{\left(1-\frac{k^2}{2}\right)K_{(k)} - E_{(k)}\right\}. \tag{2.10}$$

The components of magnetic field are calculated from $\mathbf{B} = \nabla\times\mathbf{A}$, where $\mathbf{A} = A_\varphi \mathbf{e}_\varphi$.

$$B_{r(r,z)} = -\frac{\partial A_\varphi}{\partial z} = \frac{\mu_0 I}{2\pi}\frac{z}{r}\frac{1}{\sqrt{(r+a)^2+z^2}}\left\{-K_{(k)}+\frac{r^2+a^2+z^2}{(r-a)^2+z^2}E_{(k)}\right\} \tag{2.11}$$

$$B_{z(r,z)} = \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) = \frac{\mu_0 I}{2\pi}\frac{1}{\sqrt{(r+a)^2+z^2}}\left\{K(k) + \frac{a^2-r^2-z^2}{(r-a)^2+z^2}E(k)\right\} \quad (2.12)$$

Eqs (2.11) and (2.12) indicate that the axial and radial fields are even and odd functions of z, respectively

CHAPTER 3
GENERALIZED PLANE STRESS ANALYSIS

Introduction

An axisymmetric stress analysis of superconducting magnets is formulated under assumptions that make it applicable to a region about the midplane of a long magnet. The stress analysis of superconducting magnets, as an important aspect of the overall design process, has been the subject of a number of publications.[12-19] Early analysis assumed only isotropic material properties and plane conditions, and treated only the dominant tangential stress.[12-14] These analyses were further developed to include orthotropy and solutions to the winding and thermal stresses.[15-19]

The generalized plane strain analysis provides an analytical formulation of the axisymmetric state of stress and strain in the midplane of a superconducting magnet. The present formulation takes into account the effect of the axial body force between the midplane and the end of the coil. A fundamental simplifying assumption is that the shear stress in the midplane is zero. A second assumption which leads to the present solution is that the axial strain in the midplane region of a long magnet is constant through the coil along a radius, for a fixed axial location. The applicability of this assumption for the stress analysis of superconducting magnets has been recognized previously.[20, 21]

Magnetic Stress Analysis

Stress and Strain Formulation

The magnetic stress and associated strain are the results of the distributed Lorentz forces. The equations for the magnetic stress and strain are formulated for a coil of a

superconducting magnet with a uniform current density. The coils of a superconducting magnet are treated as homogeneous, orthotropic and linear elastic material.

With the assumption of zero shear stress, the constitutive equations relating stress and strain are given by Eq. (3.1).

$$\varepsilon_\theta = \frac{\sigma_\theta}{E_\theta} - \nu_{r\theta}\frac{\sigma_r}{E_r} - \nu_{z\theta}\frac{\sigma_z}{E_z} \qquad (3.1)$$

$$\varepsilon_r = -\nu_{\theta r}\frac{\sigma_\theta}{E_\theta} + \frac{\sigma_r}{E_r} - \nu_{zr}\frac{\sigma_z}{E_z}$$

$$\varepsilon_z = -\nu_{\theta z}\frac{\sigma_\theta}{E_\theta} - \nu_{rz}\frac{\sigma_r}{E_r} + \frac{\sigma_z}{E_z}$$

Eq. (3.1) may be written in the form of the associated compliance matrix **S**.

$$\varepsilon_i = S_{ij}\sigma_j \qquad (3.2)$$

Inverting Eq. (3.2), the stress and strain are related by the stiffness matrix **C.**

$$\sigma_i = C_{ij}\,\varepsilon_j \qquad (3.3)$$

Following the traditional approach, the radial displacement u is introduced as a primary variable. The tangential and radial strains are related to the displacement as given in Eq. (3.4).

$$\varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_r = \frac{du}{dr} \qquad (3.4)$$

In Eq. (3.5), the components of the stress are written in terms of the displacement and the axial strain.

$$\sigma_\theta = C_{11}\frac{u}{r} + C_{12}\frac{du}{dr} + C_{13}\,\varepsilon_z \qquad (3.5)$$

$$\sigma_r = C_{12}\frac{u}{r} + C_{22}\frac{du}{dr} + C_{23}\,\varepsilon_z$$

$$\sigma_z = C_{13}\frac{u}{r} + C_{23}\frac{du}{dr} + C_{33}\,\varepsilon_z$$

It is assumed that the axial strain is constant throughout the coil.

$$\varepsilon_z = constant \tag{3.6}$$

In the absence of the shear stresses, the equilibrium equations at the midplane are reduced to one equation as given by Eq. (3.7).

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + rX_r = 0 \tag{3.7}$$

The radial body force X_r is due to the radial Lorentz force density given by

$$X_r = J_\theta B_{z(r)} \tag{3.8}$$

where J_θ is the current density and $B_{z(r)}$ is the axial magnetic field.

Introducing Eqs. (3.5) and (3.8) into (3.7) gives

$$C_{22}\frac{d}{dr}\left(r\frac{du}{dr}\right) - C_{11}\frac{u}{r} + (C_{23} - C_{13})\varepsilon_z = -rJ_\theta B_{z(r)}. \tag{3.9}$$

It is assumed that the radial distribution of the axial field is linear.

$$B_{z(r)} = B_c - C_0 r \tag{3.10}$$

The constants B_c and C_0 are determined from the values of the magnetic fields at the inside and outside radii. The accuracy of the linearity assumption has been previously studied in some detail.[18] This assumption is not fundamental to the method, and with a few changes in the algebra a high order polynomial for the field may be adopted.

Incorporating Eq. (3.10) in Eq. (3.9) yields Eq. (3.11), where the customary variable k is the anisotropy factor given in Eq. (3.12).

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} - k^2\frac{u}{r} = -\frac{J_\theta B_c}{C_{22}}r + \frac{J_\theta C_0}{C_{22}}r^2 - \frac{(C_{23} - C_{13})}{C_{22}}\varepsilon_z \tag{3.11}$$

$$k^2 = \frac{C_{11}}{C_{22}} \tag{3.12}$$

The general solution of Eq. (3.11) may be written in the form

$$u = D_1 r^k + D_2 r^{-k} + A_1 \varepsilon_z r + A_2 r^2 + A_3 r^3 \tag{3.13}$$

where D_1 and D_2 are integration constants, and A_1, A_2 and A_3 are constants based upon the coil's magnetic and mechanical properties.

$$A_1 = -\frac{1}{\left(1-k^2\right)} \frac{\left(C_{23} - C_{13}\right)}{C_{22}} \tag{3.14}$$

$$A_2 = -\frac{1}{\left(4-k^2\right)} \frac{J_\theta B_c}{C_{22}}$$

$$A_3 = \frac{1}{\left(9-k^2\right)} \frac{J_\theta C_0}{C_{22}}$$

By inspection, there are three singular values of the anisotropy factor k. When k has the value 1, 2, or 3, the associated constant A is replaced in the general solution by the corresponding function A given by Eq. (3.15), (3.16), or (3.17).

$$A_1(r) = -\frac{1}{2} \frac{\left(C_{23} - C_{13}\right)}{C_{22}} \ln r \tag{3.15}$$

$$A_2(r) = -\frac{1}{4} \frac{J_\theta B_c}{C_{22}} \ln r \tag{3.16}$$

$$A_3(r) = +\frac{1}{6} \frac{J_\theta C_0}{C_{22}} \ln r \tag{3.17}$$

In the remainder of the analysis, non-singular values of k are assumed.

The solution for the displacement is used in Eqs. (3.4) and (3.5) to yield expressions for the state of strain and stress as given in Eqs. (3.18) through (3.22), together with Eq. (3.6).

$$\varepsilon_\theta = D_1 r^{k-1} + D_2 r^{-k-1} + A_1 \varepsilon_z + A_2 r + A_3 r^2 \tag{3.18}$$

$$\varepsilon_r = kD_1r^{k-1} - kD_2r^{-k-1} + A_1\varepsilon_z + 2A_2r + 3A_3r^2 \tag{3.19}$$

$$\begin{aligned}\sigma_\theta = &\left(C_{11} + kC_{12}\right)D_1r^{k-1} + \left(C_{11} - kC_{12}\right)D_2r^{-k-1} \\ &+\left(C_{11} + C_{12}\right)A_1\varepsilon_z + \left(C_{11} + 2C_{12}\right)A_2r + \left(C_{11} + 3C_{12}\right)A_3r^2 + C_{13}\varepsilon_z\end{aligned} \tag{3.20}$$

$$\begin{aligned}\sigma_r = &\left(C_{12} + kC_{22}\right)D_1r^{k-1} + \left(C_{12} - kC_{22}\right)D_2r^{-k-1} \\ &+\left(C_{12} + C_{22}\right)A_1\varepsilon_z + \left(C_{12} + 2C_{22}\right)A_2r + \left(C_{12} + 3C_{22}\right)A_3r^2 + C_{23}\varepsilon_z\end{aligned} \tag{3.21}$$

$$\begin{aligned}\sigma_z = &\left(C_{13} + kC_{23}\right)D_1r^{k-1} + \left(C_{13} - kC_{23}\right)D_2r^{-k-1} \\ &+\left(C_{13} + C_{23}\right)A_1\varepsilon_z + \left(C_{13} + 2C_{23}\right)A_2r + \left(C_{13} + 3C_{23}\right)A_3r^2 + C_{33}\varepsilon_z\end{aligned} \tag{3.22}$$

These equations, Eqs. (3.18) - (3.22), together with the values of the integration constants D_1 and D_2 plus the value of the axial strain ε_z, determine the distribution of stress and strain in a given coil.

Equations for the Constants

Solutions for the integration constants D_1 and D_2 results from the boundary conditions associated with each radial layer. For a stand-alone coil with a single layer, the radial stress at the inside and outside radii will be zero. For a coil with several radial layers, additional conditions are extracted from the continuity of the radial stress and radial displacement at the interface between each layer. In this way, the number of equations match the number of integration constants. An additional equation is required to determine the unknown value of the axial strain.

The basic assumption is made that at a given axial position, ε_z is constant in each layer, and that for coils with more than one layer, the value of the axial strain is the same in all layers, including external reinforcement. This assumption is equivalent to the assumption that the coil remains a right cylinder in a neighborhood of the axial plane in which the stress is calculated.

Using the concept of a plane through the coil at a given axial location, the static equilibrium of the coil requires that the net axial stress over the plane is equal to the total applied axial load accumulated on the plane. Thus, on a plane through the coil at an axial position z,

$$\int_{a_1}^{a_{n+1}} 2\pi r \sigma_z dr = F_z \tag{3.23}$$

where a_1 and a_{n+1} are the inside and outside radii of a coil which has n distinct radial layers, and F_z is the total axial Lorentz force between the midplane and the end of the coil over all layers.

For a single layer coil the boundary conditions are

$$\sigma_r = 0 \text{ at } r = a_1 \tag{3.24}$$

$$\sigma_r = 0 \text{ at } r = a_2 \tag{3.25}$$

$$\int_{a_1}^{a_2} 2\pi r \sigma_z dr = F_z. \tag{3.26}$$

Evaluating Eq. (3.21) at the inner radius (a_1) results in

$$a_{11}D_1 + a_{12}D_2 + a_{13}\varepsilon_z = b_1 \tag{3.27}$$

where the constants are given by

$$\begin{aligned} a_{11} &= (C_{12} + kC_{22})a_1^{k-1} \\ a_{12} &= (C_{12} - kC_{22})a_1^{-k-1} \\ a_{13} &= (C_{12} + C_{22})A_1 + C_{23} \\ b_1 &= -(C_{12} + 2C_{22})A_2 a_1 - (C_{12} + 3C_{22})A_3 a_1^2. \end{aligned} \tag{3.28}$$

Evaluating Eq. (3.21) at the outer radius (a_2) leads to

$$a_{21}D_1 + a_{22}D_2 + a_{23}\varepsilon_z = b_2 \tag{3.29}$$

where the constants are given by

$$\begin{aligned}
a_{21} &= (C_{12} + kC_{22})a_2^{k-1} \\
a_{22} &= (C_{12} - kC_{22})a_2^{-k-1} \\
a_{23} &= a_{13} \\
b_2 &= -(C_{12} + 2C_{22})A_2 a_2 - (C_{12} + 3C_{22})A_3 a_2^2.
\end{aligned} \tag{3.30}$$

Integrating Eq. (3.22) in Eq. (3.26) gives

$$a_{31}D_1 + a_{32}D_2 + a_{33}\varepsilon_z = b_3 \tag{3.31}$$

where the constants are given by

$$\begin{aligned}
a_{31} &= (C_{13} + kC_{23})\frac{a_2^{k+1} - a_1^{k+1}}{k+1} \\
a_{32} &= (C_{13} - kC_{23})\frac{a_2^{-k+1} - a_1^{-k+1}}{-k+1} \\
a_{33} &= \left[(C_{13} + C_{23})A_1 + C_{33}\right]\frac{a_2^2 - a_1^2}{2} \\
b_3 &= \frac{F_z}{2\pi} - (C_{13} + 2C_{23})A_2\frac{a_2^3 - a_1^3}{3} - (C_{13} + 3C_{32})A_3\frac{a_2^4 - a_1^4}{4}.
\end{aligned} \tag{3.32}$$

The set of linear Eqs. (3.27), (3.29), and (3.31) may be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} D_1 \\ D_2 \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3.33}$$

which is solved in the standard way as given by Eq. (3.34).

$$\begin{bmatrix} D_1 \\ D_2 \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3.34}$$

For a single layer coil with a reinforcement, the boundary conditions are applied to the coil and the reinforcement.

$$\sigma_r^{(1)} = 0 \text{ at } r = a_1 \tag{3.35}$$

$$\sigma_r^{(1)} = \sigma_r^{(2)} \text{ at } r = a_2 \tag{3.36}$$

$$u_r^{(1)} = u_r^{(2)} \text{ at } r = a_2 \tag{3.37}$$

$$\sigma_r^{(2)} = 0 \text{ at } r = a_3 \tag{3.38}$$

$$\int_{a_1}^{a_3} 2\pi r \sigma_z dr = F_z \tag{3.39}$$

Evaluating Eq. (3.21) at the inner radius (a_1) results in

$$a_{11}D_1 + a_{12}D_2 + a_{15}\varepsilon_z = b_1 \tag{3.40}$$

where the constants are given by

$$\begin{aligned} a_{11} &= (C_{12} + kC_{22})a_1^{k-1} \\ a_{12} &= (C_{12} - kC_{22})a_1^{-k-1} \\ a_{15} &= (C_{12} + C_{22})A_1 + C_{23} \\ b_1 &= -(C_{12} + 2C_{22})A_2 a_1 - (C_{12} + 3C_{22})A_3 a_1^2. \end{aligned} \tag{3.41}$$

Evaluating Eq. (3.21) at the interface between the coil and the reinforcement (a_2) results in

$$a_{21}D_1 + a_{22}D_2 + a_{23}D_1' + a_{24}D_2' + a_{25}\varepsilon_z = b_2 \tag{3.42}$$

where the constants are given by

$$
\begin{aligned}
a_{21} &= (C_{12} + kC_{22})a_2^{k-1} \\
a_{22} &= (C_{12} - kC_{22})a_2^{-k-1} \\
a_{23} &= -(C'_{12} + k'C'_{22})a_2^{k'-1} \\
a_{24} &= -(C'_{12} - k'C'_{22})a_2^{-k'-1} \\
a_{25} &= \left[(C_{12} + C_{22})A_1 + C_{23}\right] - \left[(C'_{12} + C'_{22})A'_1 + C'_{23}\right] \\
b_2 &= -(C_{12} + 2C_{22})A_2 a_2 - (C_{12} + 3C_{22})A_3 a_2^2
\end{aligned} \tag{3.43}
$$

and the unprimed and primed quantities refer to the coil and the reinforcement, respectively. Evaluating Eq. (3.13) for the displacement at the interface results in

$$
a_{31}D_1 + a_{32}D_2 + a_{33}D'_1 + a_{34}D'_2 + a_{35}\varepsilon_z = b_3 \tag{3.44}
$$

where the constants are given by

$$
\begin{aligned}
a_{31} &= a_2^k \\
a_{32} &= a_2^{-k} \\
a_{33} &= -a_2^{k'} \\
a_{34} &= -a_2^{-k'} \\
a_{35} &= A_1 a_2 - A'_1 a_2 \\
b_3 &= -A_2 a_2^2 - A_3 a_2^3.
\end{aligned} \tag{3.45}
$$

Evaluating Eq. (3.21) at the outside radius of the reinforcement (a_3) results in

$$
a_{43}D'_1 + a_{44}D'_2 + a_{45}\varepsilon_z = 0 \tag{3.46}
$$

where the constants are given by

$$
\begin{aligned}
a_{43} &= (C'_{12} + k'C'_{22})a_3^{k'-1} \\
a_{44} &= (C'_{12} - k'C'_{22})a_3^{-k'-1} \\
a_{45} &= (C'_{12} + C'_{22})A'_1 + C'_{23}.
\end{aligned} \tag{3.47}
$$

Integrating Eq. (3.22) through coil and reinforcement in Eq. (3.39) results in

$$a_{51}D_1 + a_{52}D_2 + a_{53}D_1' + a_{54}D_2' + a_{55}\varepsilon_z = b_5 \tag{3.48}$$

where the constants are given by

$$
\begin{aligned}
a_{51} &= \left(C_{13} + kC_{23}\right)\frac{a_2^{k+1} - a_1^{k+1}}{k+1} \\
a_{52} &= \left(C_{13} - kC_{23}\right)\frac{a_2^{-k+1} - a_1^{-k+1}}{-k+1} \\
a_{53} &= \left(C_{13}' + k'C_{23}'\right)\frac{a_3^{k'+1} - a_2^{k'+1}}{k'+1} \\
a_{54} &= \left(C_{13}' - k'C_{23}'\right)\frac{a_3^{-k'+1} - a_2^{-k'+1}}{-k'+1} \\
a_{55} &= \left[(C_{13} + C_{23})A_1 + C_{33}\right]\frac{a_2^2 - a_1^2}{2} + \left[(C_{13}' + C_{23}')A_1' + C_{33}'\right]\frac{a_3^2 - a_2^2}{2} \\
b_5 &= -(C_{13} + 2C_{23})A_2\frac{a_2^3 - a_1^3}{3} - (C_{13} + 3C_{32})A_3\frac{a_2^4 - a_1^4}{4} + \frac{F_z}{2\pi}.
\end{aligned} \tag{3.49}
$$

The set of linear Eqs. (3.40), (3.42), (3.44), (3.46), and (3.48) may be written in matrix form

$$
\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
0 & 0 & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{bmatrix}
\begin{bmatrix} D_1 \\ D_2 \\ D_1' \\ D_2' \\ \varepsilon_z \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \tag{3.50}
$$

which is solved as expressed by Eq. (3.51).

$$
\begin{bmatrix} D_1 \\ D_2 \\ D_1' \\ D_2' \\ \varepsilon_z \end{bmatrix}
=
\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & a_{15} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\
0 & 0 & a_{43} & a_{44} & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55}
\end{bmatrix}^{-1}
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \tag{3.51}
$$

For a general multilayer coil (n layers) the boundary conditions are applied to each layer.

$$\sigma_r^{(1)} = 0 \text{ at } r = a_1 \tag{3.52}$$

$$\sigma_r^{(j)} = \sigma_r^{(j+1)} \text{ at } r = a_{j+1};\ j = 1, n-1 \tag{3.53}$$

$$u_r^{(j)} = u_r^{(j+1)} \text{ at } r = a_{j+1};\ j = 1, n-1 \tag{3.54}$$

$$\sigma_r^{(n)} = 0 \text{ at } r = a_{n+1} \tag{3.55}$$

The axial force equilibrium also applies to all layers.

$$\int_{a_1}^{a_{n+1}} 2\pi r \sigma_z dr = F_z \tag{3.56}$$

The derivation of the equations for the constants proceeds as in the previous cases. The coefficient matrix for the system of $2n$+1 equations and unknowns is given by Eq. (3.57).

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & \cdots & & & a_{1\,2n+1} \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & \cdots & & & a_{2\,2n+1} \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & \cdots & & & a_{3\,2n+1} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} & 0 & \cdots & & & a_{4\,2n+1} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} & 0 & \cdots & & & a_{5\,2n+1} \\ \vdots & \vdots & & & & & & & & & \vdots \\ 0 & 0 & \cdots & & & & 0 & a_{2n\,2n-1} & & a_{2n\,2n} & a_{2n\,2n+1} \\ a_{2n+1\,1} & a_{2n+1\,2} & \cdots & & & & & & & a_{2n+1\,2n} & a_{2n+1\,2n+1} \end{bmatrix} \tag{3.57}$$

The solution for this system of equations is obtained by matrix inversion.

Thermal Stresses Analysis

The thermal stress and associated strain are the result of anisotropy in thermal contraction coefficients within a coil, or differences in thermal contraction coefficients between the layers including reinforcement. The equations for thermal stress and strain are formulated for a coil of a superconducting magnet.

With the assumption of zero shear, the stress is related to the mechanical component of the total strain by the stiffness matrix **C**.

$$\sigma_i = C_{ij}\varepsilon_j \tag{3.58}$$

It is recognized, in dealing with change in temperature, that the total strain within a body is the sum of strains produced by mechanical stresses and strains associated with thermal expansion or contraction. The radial displacement u, that is introduced as a primary variable in the analysis, is directly related to the total strains in the tangential and radial directions. The mechanical strains are given by

$$\begin{aligned} \varepsilon_\theta &= \frac{u}{r} - \alpha_\theta \Delta T \\ \varepsilon_r &= \frac{du}{dr} - \alpha_r \Delta T \\ \varepsilon_z &= \varepsilon_z^{tot} - \alpha_z \Delta T \end{aligned} \tag{3.59}$$

where ε_z^{tot} is the total axial strain. Introducing Eq.(3.59) into Eq. (3.58) gives the stress in terms of the displacement and the total axial strain. The generalized plane strain assumption applies to the total axial strain.

$$\varepsilon_z^{tot} = constant \tag{3.60}$$

In the absence of the shear stresses and body forces, the equilibrium equations are reduced to one equation as defined by Eq. (3.61).

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0 \tag{3.61}$$

Introducing Eqs. (3.59) into Eq. (3.61) yields

$$\begin{aligned} r\frac{d^2u}{dr^2} + \frac{du}{dr} - k^2\frac{u}{r} = &-\frac{C_{23}-C_{13}}{C_{22}}\varepsilon_z^{tot} \\ &+ \frac{C_{12}-C_{11}}{C_{22}}\alpha_\theta\Delta T + \frac{C_{22}-C_{12}}{C_{22}}\alpha_r\Delta T + \frac{C_{23}-C_{13}}{C_{22}}\alpha_z\Delta T \end{aligned} \tag{3.62}$$

where the anisotropy factor k is given by

$$k^2 = \frac{C_{11}}{C_{22}}. \tag{3.63}$$

The general solution of Eq. (3.62) may be written in the form

$$u = D_1 r^k + D_2 r^{-k} + A_1 \varepsilon_z^{tot} r + A_2 r \tag{3.64}$$

where D_1 and D_2 are integration constants, and A_1 and A_2 are constants based upon the coil's mechanical and thermal properties.

$$A_1 = -\frac{1}{1-k^2}\frac{C_{23}-C_{13}}{C_{22}} \tag{3.65}$$
$$A_2 = \frac{1}{1-k^2}\frac{(C_{12}-C_{11})\alpha_\theta \Delta T + (C_{22}-C_{12})\alpha_r \Delta T + (C_{23}-C_{13})\alpha_z \Delta T}{C_{22}}$$

For an isotropic material ($k = 1$), the right hand side of Eq. (3.62) vanishes, and hence, A_1 and A_2 become zero.

The solution for the displacement introduced through Eq. (3.59) yields the state of stress and strain as given in Eqs. (3.66) - (3.70), together with Eq. (3.60).

$$\varepsilon_\theta = D_1 r^{k-1} + D_2 r^{-k-1} + A_1 \varepsilon_z^{tot} + A_2 - \alpha_\theta \Delta T \tag{3.66}$$

$$\varepsilon_r = D_1 k r^{k-1} - D_2 k r^{-k-1} + A_1 \varepsilon_z^{tot} + A_2 - \alpha_r \Delta T \tag{3.67}$$

$$\begin{aligned}\sigma_\theta &= (C_{11} + kC_{12}) D_1 r^{k-1} + (C_{11} - kC_{12}) D_2 r^{-k-1} \\ &+ (C_{11} + C_{12}) A_1 \varepsilon_z^{tot} + (C_{11} + C_{12}) A_2 + C_{13} \varepsilon_z^{tot} \\ &- C_{11}\alpha_\theta \Delta T - C_{12}\alpha_r \Delta T - C_{13}\alpha_z \Delta T\end{aligned} \tag{3.68}$$

$$\begin{aligned}\sigma_r &= (C_{12} + kC_{22}) D_1 r^{k-1} + (C_{12} - kC_{22}) D_2 r^{-k-1} \\ &+ (C_{12} + C_{22}) A_1 \varepsilon_z^{tot} + (C_{12} + C_{22}) A_2 + C_{23} \varepsilon_z^{tot} \\ &- C_{12}\alpha_\theta \Delta T - C_{22}\alpha_r \Delta T - C_{23}\alpha_z \Delta T\end{aligned} \tag{3.69}$$

$$\sigma_z = (C_{13} + kC_{23})D_1 r^{k-1} + (C_{13} - kC_{23})D_2 r^{-k-1}$$
$$+(C_{13} + C_{23})A_1\varepsilon_z^{tot} + (C_{13} + C_{23})A_2 + C_{33}\varepsilon_z^{tot}$$
$$-C_{13}\alpha_\theta \Delta T - C_{23}\alpha_r \Delta T - C_{33}\alpha_z \Delta T \tag{3.70}$$

These equations together with the value of the integration constants D_1 and D_2 and the value of the constant axial strain ε_z^{tot}, determine the stress and strain in the coil. The equations for the integration constants and the axial strain are given in Appendix A.

CHAPTER 4
GREEN'S FUNCTIONS SOLUTION

Introduction

A full axisymmetric stress analysis in terms of Green's functions for an elastic isotropic coil of a superconducting magnet is presented. This analysis is applicable throughout the coil and includes the shear stress in addition to the normal stresses. As the size and field strength of high field magnets increase, the knowledge of stress distribution, especially shear stress, beyond the midplane of the coil becomes increasingly important. A limited number of publications on the approximation of the three-dimensional problem are available in the literature.[22-25] Some solutions are obtained by neglecting shear throughout the coil.[22,23] Other solutions are provided based upon numerical techniques[24] or power series expansion of the fields and displacements.[25] The direct analytical solutions are formulated for infinite or semi-infinite domains that are not appropriate for a finite domain such as a magnet.[26,27] It is desirable to develop a full stress analysis for a superconducting magnet irrespective of the type of field or its distribution within the coil. As a first approach a coil with isotropic material properties is assumed.

In the following sections stresses are formulated using stress functions. Stress functions are solved in terms of Green's functions by exploiting finite integral transforms, e.g., finite Hankel transform.

Fundamental Equations for the Stress Functions

Consider an elastic isotropic coil with inside radius of a, outside radius of b, and length of $2L$ as shown in Fig. 4.1. For an axisymmetric distribution of Lorentz force density, $\mathbf{X}(r,z)$, the equilibrium equation may be written as

$$\nabla.\boldsymbol{\sigma}+\mathbf{X}=0 \tag{4.1}$$

where $\boldsymbol{\sigma}$ is the stress tensor.

The constitutive equation relating the stress tensor to the strain tensor $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\sigma}=\lambda\, tr(\boldsymbol{\varepsilon})\mathbf{I}+2\mu\boldsymbol{\varepsilon} \tag{4.2}$$

where λ and μ are (Lamé's) elastic coefficients and $tr(\boldsymbol{\varepsilon})$ is the trace of the strain tensor.

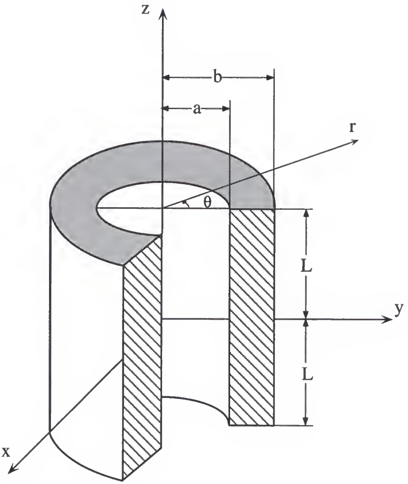

Figure 4.1 Schematic diagram representing one coil of a magnet.

The displacement vector $\mathbf{u}_{(r,z)}$ is related to the strain tensor as described by Eq. (4.3).

$$\boldsymbol{\varepsilon}=\frac{1}{2}\left[\nabla\mathbf{u}+\nabla^{T}\mathbf{u}\right] \tag{4.3}$$

Incorporating Eq. (4.3) into Eq. (4.2) yields an equation for the stress tensor in terms of displacement vector.

$$\sigma = \lambda(\nabla.\mathbf{u})\mathbf{I} + \mu\left[\nabla\mathbf{u} + \nabla^{\mathrm{T}}\mathbf{u}\right] \tag{4.4}$$

Introducing Eq. (4.4) into Eq. (4.1) results in a partial differential equation for displacement vector.

$$(\lambda + \mu)\nabla(\nabla.\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{X} = 0 \tag{4.5}$$

From the Helmholtz theorem, any vector satisfying Eq. (4.5) may be resolved into a sum of a gradient and a curl

$$\mathbf{u} = \nabla\phi + \nabla\times\mathbf{A} \tag{4.6}$$

where $\phi_{(r,z)}$ is a scalar potential and $\mathbf{A}_{(r,z)}$ is a vector potential such that $\nabla.\mathbf{A} = 0$. Incorporating the displacement vector from Eq. (4.6) into Eq. (4.5) yields an equation in terms of potential functions ϕ and $\mathbf{A}$.

$$(\lambda + 2\mu)\nabla(\nabla^2\phi) + \mu\nabla\times(\nabla^2\mathbf{A}) + \mathbf{X} = 0 \tag{4.7}$$

It is admissible[26] to express the independent potential functions ϕ and $\mathbf{A}$ as

$$\mathbf{A} = \alpha\nabla\times\Psi \qquad\qquad \phi = \beta\nabla.\Psi \tag{4.8}$$

where α and β are arbitrary constants and components of the vector Ψ are the stress functions. Introducing Eq. (4.8) into Eq. (4.7) leads to a partial differential equation for vector Ψ.

$$\beta(\lambda + 2\mu)\nabla^4\Psi + [\beta(\lambda + 2\mu) + \mu\alpha]\nabla\times[\nabla\times(\nabla^2\Psi)] + \mathbf{X} = 0 \tag{4.9}$$

In order to simplify Eq. (4.9), we may choose arbitrary constants α and β as $-\frac{1}{\mu}$ and $\frac{1}{\lambda + 2\mu}$, respectively. Thus, Eq. (4.9) reduces to

$$\nabla^4 \boldsymbol{\Psi} + \mathbf{X} = 0. \tag{4.10}$$

In cylindrical coordinates (r, θ, z), Laplacian of vector $\boldsymbol{\Psi}$ is given by

$$\nabla^2 \boldsymbol{\Psi} = \left[\nabla^2 \Psi_r - \frac{\Psi_r}{r^2} - \frac{2}{r^2}\frac{\partial \Psi_\theta}{\partial \theta}\right]\mathbf{e}_r + \left[\nabla^2 \Psi_\theta - \frac{\Psi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \Psi_r}{\partial \theta}\right]\mathbf{e}_\theta + \nabla^2 \Psi_z \mathbf{e}_z \tag{4.11}$$

where the Laplacian (∇^2) of a scalar is represented by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

Applying Eq. (4.11) to Eq. (4.10) results in three partial differential equations for stress functions.*

$$\left(\nabla^2 - \frac{1}{r^2}\right)^2 \Psi_r - \frac{4}{r^4}\frac{\partial^2 \Psi_r}{\partial \theta^2} - \frac{4}{r^2}\left(\nabla^2 - \frac{1}{r^2}\right)\frac{\partial \Psi_\theta}{\partial \theta} + X_r = 0 \tag{4.12}$$

$$\left(\nabla^2 - \frac{1}{r^2}\right)^2 \Psi_\theta - \frac{4}{r^4}\frac{\partial^2 \Psi_\theta}{\partial \theta^2} + \frac{4}{r^2}\left(\nabla^2 - \frac{1}{r^2}\right)\frac{\partial \Psi_r}{\partial \theta} + X_\theta = 0$$

$$\nabla^4 \Psi_z + X_z = 0$$

The geometry is axisymmetric in nature and for an axisymmetric system, the partial derivative with respect to tangential direction is zero, $\frac{\partial}{\partial \theta} = 0$. Thus, the three partial differential equations in Eq. (4.12) will reduce to three uncoupled partial differential equations given by Eqs. (4.13) - (4.15).

$$\left(\nabla^2 - \frac{1}{r^2}\right)^2 \Psi_r + X_r = 0 \tag{4.13}$$

$$\left(\nabla^2 - \frac{1}{r^2}\right)^2 \Psi_\theta + X_\theta = 0 \tag{4.14}$$

---

* See Appendix B for details of the derivation.

$$\nabla^4 \Psi_z + X_z = 0 \tag{4.15}$$

The driving body force in a magnet is due to the Lorentz force density. The Lorentz force is a function of r and z, and is related to the magnetic field and current density by

$$\mathbf{X} = \mathbf{J} \times \mathbf{B} \tag{4.16}$$

where **J** and **B** are the current density and the magnetic field vectors, respectively. For an axisymmetric magnet $\mathbf{J} = J_\theta \mathbf{e}_\theta$ and $\mathbf{B} = B_r \mathbf{e}_r + B_z \mathbf{e}_z$. The vector product of the **J** and **B** leads to

$$X_r = J_\theta B_z \qquad X_\theta = 0 \qquad X_z = -J_\theta B_r. \tag{4.17}$$

Thus, in the absence of a tangential magnetic body force, Eq. (4.14) yields $\Psi_\theta = 0$, which results in tangential displacement u_θ to be zero. Solutions to Eqs. (4.13) and (4.15) will determine the stress functions and hence the stresses. In order to solve these equations, we need to exploit the finite Hankel transform.

Finite Hankel Transform

In contrast to the Hankel transform that is available in the literature corresponding to an infinite domain, the finite Hankel transform is a rare commodity.[28, 29] The ones that are available are mainly used for an interval starting at the origin. Hence, they carry only the Bessel function of the first kind. For the case of a coil (a hollow cylinder), the radial interval starts at a point greater than zero, thus, a suitable finite Hankel transform must contain the Bessel function of the first kind and the Bessel function of the second kind. A detailed derivation of the finite Hankel transform for an interval appropriate for a coil is carried out in this section.

Derivation

A function $I_{f(\alpha)}$, as expressed in Eq. (4.18), defines an integral transform of an arbitrary function $f_{(x)}$ by a kernel $K_{(\alpha,x)}$.

$$I_{f}(\alpha) = \int_{0}^{\infty} f(x) K(\alpha, x) dx \tag{4.18}$$

The method of integral transform may under some circumstances be applied to the solution of boundary value and initial value problems in mathematical physics. In problems in which one of the independent variables, say x, has a range of $(0,\infty)$ the use of an integral transform of the type $\int_{0}^{\infty} f(x)K(\alpha,x)dx$ will reduce a partial differential equation in n independent variables to n-1 independent variables thus, reducing the complexity of the problem. In some instances successive operations of this type will ultimately reduce the partial differential equation into an ordinary differential equation. It is possible to extend the method of integral transforms to a finite interval in which the field of variation of the independent variable is now $[a,b]$ where both a and b are real and finite.

A Fourier Bessel kernel $K_n(\zeta_i, x)$ based upon a solution of a Bessel equation, $y'' + \frac{1}{r}y' + \left(\zeta^2 - \frac{n^2}{r^2}\right)y = 0$, with homogeneous Dirichlet conditions in the finite interval $[a,b]$ where a is greater than zero, is expressed by

$$K_n(\zeta_i, x) = \left[J_n(\zeta_i x)Y_n(\zeta_i b) - J_n(\zeta_i b)Y_n(\zeta_i x)\right]. \tag{4.19}$$

Here, ζ_i is a root of the transcendental equation given by Eq. (4.20).

$$J_n(\zeta_i a)Y_n(\zeta_i b) - J_n(\zeta_i b)Y_n(\zeta_i a) = 0 \tag{4.20}$$

The kernel $K_n(\zeta_i, x)$ is orthogonal* in the interval $[a,b]$ with respect to a weighting function $r(x) = x$. Thus, we may write

$$\int_{a}^{b} xK_n(\zeta_i, x)K_n(\zeta_j, x)dx = \begin{cases} 0, & \zeta_i \neq \zeta_j \\ \|K_n(\zeta_i, x)\|^2, & \zeta_i = \zeta_j \end{cases} \tag{4.21}$$

* See Eq. (C.18) of Appendix C.

where both ζ_i and ζ_j are roots of Eq. (4.20) and $\|K_{n(\zeta_i,x)}\|$ is the norm of the $K_{n(\zeta_i,x)}$ that is expressed* by Eq. (4.22).

$$\|K_{n(\zeta_i,x)}\| = \frac{\sqrt{2}}{\pi\zeta_i}\left[1-\left(\frac{J_{n(\zeta_i b)}}{J_{n(\zeta_i a)}}\right)^2\right]^{1/2} \tag{4.22}$$

From the theory of Bessel functions, if $f_{(x)}$ satisfies the Dirichlet conditions in the interval $[a,b]$, then the series

$$\sum A_{(\zeta_i)}\left[J_{n(\zeta_i x)}Y_{n(\zeta_i b)} - J_{n(\zeta_i b)}Y_{n(\zeta_i x)}\right]$$

converges to the sum of $\frac{1}{2}\left[f_{(x+0)}+f_{(x-0)}\right]$. Thus, if $f_{(x)}$ is continuous at the point x the series converges to $f_{(x)}$.

$$f_{(x)} = \sum_i A_{(\zeta_i)}K_{n(\zeta_i,x)} \tag{4.23}$$

In Eq. (4.23) $A_{(\zeta_i)}$ is an arbitrary function of ζ_i, $K_{n(\zeta_i,x)}$ is the Fourier Bessel kernel, and the summation is taken over all the positive roots of ζ_i.

In order to obtain an expression for the $A_{(\zeta_i)}$ in Eq. (4.23) we make use of the orthogonality property of $K_{n(\zeta_i,x)}$. Multiplying both sides of the Eq. (4.23) by the term $xK_{n(\zeta_j,x)}$, integrating over the interval $[a,b]$, and assuming that term by term integration is permissible, yields

$$\int_a^b xf_{(x)}K_{n(\zeta_j,x)}dx = \sum_i A_{(\zeta_i)}\int_a^b xK_{n(\zeta_i,x)}K_{n(\zeta_j,x)}dx. \tag{4.24}$$

All terms on the right hand side of Eq. (4.24) vanish except the term corresponding to $i=j$. Thus, Eq. (4.24) is reduced to Eq. (4.25).

* See Eq. (C.38) of Appendix C.

$$\int_a^b xf_{(x)}K_{n(\zeta_i,x)}dx = A_{(\zeta_i)}\|K_{n(\zeta_i,x)}\|^2 \tag{4.25}$$

Eq. (4.25) is the foundation for the finite Hankel transfer theorem. If $f_{(x)}$ satisfies Dirichlet conditions in an interval $[a,b]$, then its finite Hankel transform of order n, denoted by $\bar{f}_{(\zeta_i)}$, is defined by

$$\Re_n[f_{(x)}] = \bar{f}_{(\zeta_i)} = \int_a^b xf_{(x)}K_{n(\zeta_i,x)}dx \tag{4.26}$$

where ζ_i is a root of the Eq. (4.20), $\Re_n[f_{(x)}]$ is a linear functional operator, and $K_{n(\zeta_i,x)}$ is the Fourier Bessel kernel given by Eq. (4.19). By using Eqs.(4.25) and (4.26) we may write the arbitrary function, $A_{(\zeta_i)}$, in terms of the finite Hankel transform of $f_{(x)}$.

$$A_{(\zeta_i)} = \|K_{(\zeta_i,x)}\|^{-2}\bar{f}_{(\zeta_i)} \tag{4.27}$$

Incorporating Eqs. (4.27) and (4.22) into Eq. (4.23) yields the following inversion theorem. If $f_{(x)}$ satisfies the Dirichlet conditions in an interval $[a,b]$ and if its finite Hankel transform is $\bar{f}_{(\zeta_i)}$ then at each point in that interval $f_{(x)}$ may be written as

$$f_{(x)} = \Re_n^{-1}[\bar{f}_{(\zeta_i)}] = \frac{\pi^2}{2}\sum_i \frac{\zeta_i^2 J_{n(\zeta_i a)}^2}{J_{n(\zeta_i a)}^2 - J_{n(\zeta_i b)}^2}\bar{f}_{(\zeta_i)}K_{n(\zeta_i,x)} \tag{4.28}$$

where the summation is extended over all the positive roots of ζ_i.

Finite Hankel Transforms of the Derivatives

Suppose that $\bar{f}_{(\zeta_i)}$ is the finite Hankel transform of $f_{(x)}$. The finite Hankel transform of the one-dimensional Laplacian of $f_{(x)}$ in cylindrical coordinates, $\nabla^2 f = \frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx}$, is then expressed by Eq. (4.29).

$$\Re_n[\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx}] = \int_a^b x\left[\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right]K_{n(\zeta_i,x)}dx \tag{4.29}$$

Integration by parts of the right hand side of the Eq. (4.29) gives

$$\int_a^b x\left[\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right]K_{n(\zeta_i,x)}dx = x\frac{df}{dx}K_{n(\zeta_i,x)}\Big|_a^b - xf\frac{\partial K_{n(\zeta_i,x)}}{\partial x}\Big|_a^b + \int_a^b f\frac{\partial}{\partial x}\left[x\frac{\partial K_{n(\zeta_i,x)}}{\partial x}\right]dx. \tag{4.30}$$

The kernel $K_{n(\zeta_i,x)}$ in Eq. (4.30) is derived from Bessel functions of the first and second kind, therefore, it satisfies the Bessel equation.

$$x\frac{\partial}{\partial x}\left(x\frac{\partial K_{n(\zeta_i,x)}}{\partial x}\right)+\left(\zeta_i^2x^2-n^2\right)K_{n(\zeta_i,x)}=0 \tag{4.31}$$

Introducing Eq. (4.31) into Eq. (4.30) yields

$$\int_a^b x\left[\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right]K_{n(\zeta_i,x)}dx = x\frac{df}{dx}K_{n(\zeta_i,x)}\Big|_a^b - xf\frac{\partial K_{n(\zeta_i,x)}}{\partial x}\Big|_a^b - \int_a^b f\left(\zeta_i^2x-\frac{n^2}{x}\right)K_{n(\zeta_i,x)}dx \tag{4.32}$$

or

$$\int_a^b x\left[\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f\right]K_{n(\zeta_i,x)}dx = x\frac{df}{dx}K_{n(\zeta_i,x)}\Big|_a^b - xf\frac{\partial K_{n(\zeta_i,x)}}{\partial x}\Big|_a^b - \zeta_i^2\int_a^b xf_{(x)}K_{n(\zeta_i,x)}dx. \tag{4.33}$$

The first term on the right hand side of Eq. (4.33) vanishes at both upper and lower limits. The integral part of the last term is the finite Hankel transform of $f_{(x)}$. Hence, Eq. (4.33) takes the following form.

$$\int_a^b x\left[\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f\right]K_{n(\zeta_i,x)}dx = -xf\frac{\partial K_{n(\zeta_i,x)}}{\partial x}\Big|_a^b - \zeta_i^2\bar{f}_{(\zeta_i)} \tag{4.34}$$

Now, it is shown that*

$$xf_{(x)}\frac{\partial K_{n}(\zeta_i,x)}{\partial x}\bigg|_a^b = -\frac{2}{\pi}\left(f_{(b)} - \frac{J_n(\zeta_i b)}{J_n(\zeta_i a)}f_{(a)}\right). \tag{4.35}$$

The substitution of Eq. (4.35) into Eq. (4.34) yields an expression for the finite Hankel transforms of the derivatives of $f_{(x)}$.

$$\Re_n\left[\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f\right] = \frac{2}{\pi}\left(f_{(b)} - \frac{J_n(\zeta_i b)}{J_n(\zeta_i a)}f_{(a)}\right) - \zeta_i^2\Re_n\left[f_{(x)}\right] \tag{4.36}$$

Green's Functions

The partial differential equation for radial and axial stress functions are given by Eqs. (4.13) and (4.15). Solutions to these equations may be obtained by variety of methods. The present work provides the solutions in the form of Green's functions. The usage of Green's functions allows us to obtain a solution irrespective of the field's type or its distribution within the coil. Moreover, the same Green's functions may be used in order to obtain a solution for the thermal stresses.

In the next two sections, solutions to Eqs. (4.13) and (4.15), with Dirichlet homogeneous conditions are obtained in the form of radial and axial Green's functions, respectively.

Radial Green's Function

In order to find a solution for Eq. (4.13), two finite integral transforms, the finite Hankel transform in r and the finite Fourier cosine transform in z, are exploited. Applying the finite Hankel transform (of order one) to each term of the Eq. (4.13) yields

$$\Re_1\left[\left(\nabla^2 - \frac{1}{r^2}\right)^2 \Psi_r\right] = -\Re_1[X_r]. \tag{4.37}$$

* See Eqs. (C.35) and (C.36) of Appendix C.

By using the property of the finite Hankel transform (for derivatives), Eq. (4.37) reduces to

$$\left(-\zeta_{1i}^2+\frac{\partial^2}{\partial z^2}\right)^2\overline{\Psi}_{r(\zeta_{1i},z)}=-\overline{X}_{r(\zeta_{1i},z)} \tag{4.38}$$

where $\overline{\Psi}_{r(\zeta_{1i},z)}$ and $\overline{X}_{r(\zeta_{1i},z)}$ are the finite Hankel transforms of the radial stress function and radial body force, respectively.

$$\overline{\Psi}_{r(\zeta_{1i},z)}=\int_a^b r\Psi_{r(r,z)}K_{1(\zeta_{1i},r)}dr \tag{4.39}$$

$$\overline{X}_{r(\zeta_{1i},z)}=\int_a^b rX_{r(r,z)}K_{1(\zeta_{1i},r)}dr \tag{4.40}$$

The Fourier Bessel kernel $K_{1(\zeta_{1i},r)}$ in Eqs. (4.39) and (4.40) is expressed by

$$K_{1(\zeta_{1i},r)}=\left[J_{1(\zeta_{1i}r)}Y_{1(\zeta_{1i}b)}-J_{1(\zeta_{1i}b)}Y_{1(\zeta_{1i}r)}\right] \tag{4.41}$$

where ζ_{1i} is a root of the corresponding transcendental equation.

$$J_{1(\zeta_{1i}a)}Y_{1(\zeta_{1i}b)}-J_{1(\zeta_{1i}b)}Y_{1(\zeta_{1i}a)}=0 \tag{4.42}$$

An additional transform (in the axial direction) is imperative for solving Eq. (4.38). Considering the radial body force is an even function of z, an appropriate transform is the finite Fourier cosine transform. By introducing the finite Fourier cosine transform

$$\Im_C\left[\left(-\zeta_{1i}^2+\frac{\partial^2}{\partial z^2}\right)^2\overline{\Psi}_{r(\zeta_{1i},z)}\right]=-\Im_C\left[\overline{X}_{r(\zeta_{1i},z)}\right], \tag{4.43}$$

the ordinary differential equation, Eq. (4.38), is converted into an algebraic equation expressed by

$$\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)^2\overline{\overline{\Psi}}_{r(\zeta_{1i},n)}=-\overline{\overline{X}}_{r(\zeta_{1i},n)} \tag{4.44}$$

where n is an integer number. Functions $\overline{\overline{\Psi}}_{r(\zeta_{1i},n)}$ and $\overline{\overline{X}}_{r(\zeta_{1i},n)}$ are the finite Fourier cosine transforms of $\overline{\Psi}_{r(\zeta_{1i},z)}$ and $\overline{X}_{r(\zeta_{1i},z)}$, respectively.

$$\overline{\overline{\Psi}}_{r(\zeta_{1i},n)} = \int_{-L}^{L} \overline{\Psi}_{r(\zeta_{1i},z)} \cos\frac{n\pi z}{L} dz \tag{4.45}$$

$$\overline{\overline{X}}_{r(\zeta_{1i},n)} = \int_{-L}^{L} \overline{X}_{r(\zeta_{1i},z)} \cos\frac{n\pi z}{L} dz \tag{4.46}$$

From Eq. (4.44), $\overline{\overline{\Psi}}_{r(\zeta_{1i},n)}$ may be written in terms of $\overline{\overline{X}}_{r(\zeta_{1i},n)}$.

$$\overline{\overline{\Psi}}_{r(\zeta_{1i},n)} = \frac{-L^4}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2} \overline{\overline{X}}_{r(\zeta_{1i},n)} \tag{4.47}$$

The inverse finite Fourier cosine transform of $\overline{\overline{\Psi}}_{r(\zeta_{1i},n)}$ is defined by Eq. (4.48).

$$\overline{\Psi}_{r(\zeta_{1i},z)} = \Im_C^{-1}\left[\overline{\overline{\Psi}}_{r(\zeta_{1i},n)}\right] = \frac{1}{2L}\overline{\overline{\Psi}}_{r(\zeta_{1i},0)} + \frac{1}{L}\sum_{n=1}^{\infty}\overline{\overline{\Psi}}_{r(\zeta_{1i},n)}\cos\frac{n\pi z}{L} \tag{4.48}$$

Incorporating Eq. (4.47) into Eq. (4.48) yields

$$\overline{\Psi}_{r(\zeta_{1i},z)} = -\frac{1}{2L\zeta_{1i}^4}\overline{\overline{X}}_{r(\zeta_{1i},0)} - L^3\sum_{n=1}^{\infty}\frac{1}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2}\overline{\overline{X}}_{r(\zeta_{1i},n)}\cos\frac{n\pi z}{L}. \tag{4.49}$$

The inverse finite Hankel transform of $\overline{\Psi}_{r(\zeta_{1i},z)}$ is given by Eq. (4.50), where $K_{1(\zeta_{1i},r)}$ is the same Fourier Bessel kernel expressed by Eq. (4.41).

$$\Psi_{r(r,z)} = \Re_1^{-1}\left[\overline{\Psi}_{r(\zeta_{1i},z)}\right] = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\zeta_{1i}^2 J_1^2{}_{(\zeta_{1i}a)}}{J_1^2{}_{(\zeta_{1i}a)} - J_1^2{}_{(\zeta_{1i}b)}}\overline{\Psi}_{r(\zeta_{1i},z)}K_{1(\zeta_{1i},r)} \tag{4.50}$$

The substitution of $\overline{\Psi}_{r(\zeta_{1i},z)}$ from Eq. (4.49) into Eq. (4.50) yields an equation for the radial stress function.

$$\Psi_{r(r,z)} = -\frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\zeta_{1i}^2 J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)}\left[\frac{1}{2L\zeta_{1i}^4}\overline{\overline{X}}_{r(\zeta_{1i},0)} + L^3\sum_{n=1}^{\infty}\frac{1}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2}\overline{\overline{X}}_{r(\zeta_{1i},n)}\cos\frac{n\pi z}{L}\right]K_1(\zeta_{1i},r) \tag{4.51}$$

Introducing Eqs. (4.40) and (4.46) into Eq. (4.51) gives the final solution of the radial stress function $\Psi_{r(r,z)}$ in terms of the radial body force $X_{r(r,z)}$.

$$\Psi_{r(r,z)} = -\frac{\pi^2}{2}\sum_{i=1}^{\infty}\left\{\frac{1}{2L\zeta_{1i}^4}\int_{-L}^{L}\int_{a}^{b} r' X_{r(r',z')} K_1(\zeta_{1i},r') dr'dz' + L^3\sum_{n=1}^{\infty}\left[\frac{1}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2}\cos\frac{n\pi z}{L}\int_{-L}^{L}\int_{a}^{b} r' X_{r(r',z')} K_1(\zeta_{1i},r')\cos\frac{n\pi z'}{L} dr'dz'\right]\frac{\zeta_{1i}^2 J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)}K_1(\zeta_{1i},r)\right\} \tag{4.52}$$

By interchanging integrals with summations, we can write Eq. (4.52) in the form

$$\Psi_{r(r,z)} = -\int_{-L}^{L}\int_{a}^{b}\sum_{i=1}^{\infty}\left\{\frac{\pi^2}{2}\frac{\zeta_{1i}^2 J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)} r' K_1(\zeta_{1i},r) K_1(\zeta_{1i},r')\left[\frac{1}{2L\zeta_{1i}^4} + L^3\sum_{n=1}^{\infty}\frac{1}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2}\cos\frac{n\pi z'}{L}\cos\frac{n\pi z}{L}\right]\right\} X_{r(r',z')} dr'dz' \tag{4.53}$$

or

$$\Psi_{r(r,z)} = \int_{-L}^{L}\int_{a}^{b} X_{r(r',z')} G_{r(r,r',z,z')} dr'dz' \tag{4.54}$$

where $G_{r(r,r',z,z')}$ is the radial Green's function defined by Eq. (4.55).

$$G_{r(r,r',z,z')} = -\sum_{i=1}^{\infty}\left\{\frac{\pi^2}{2}\frac{\zeta_{1i}^2 J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)} r' K_1(\zeta_{1i},r) K_1(\zeta_{1i},r')\left[\frac{1}{2L\zeta_{1i}^4} + L^3\sum_{n=1}^{\infty}\frac{1}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2}\cos\frac{n\pi z'}{L}\cos\frac{n\pi z}{L}\right]\right\} \tag{4.55}$$

Axial Green's Function

The axial Green's function is obtained by solving the partial differential equation for the axial stress function, Eq. (4.15). In order to solve Eq. (4.15), the same finite integral transform (finite Hankel transform) with a different order in r is used. However, since the axial body force is an even function of z, the finite Fourier sine transform (instead of finite Fourier cosine transform) is utilized in the axial direction.

Applying the finite Hankel transform of order zero to each term of the Eq. (4.15) yields

$$\Re_0[\nabla^4 \Psi_z] = -\Re_0[X_z]. \tag{4.56}$$

By using the property of the finite Hankel transform (for derivatives), Eq. (4.56) reduces to

$$\left(-\zeta_{0i}^2 + \frac{\partial^2}{\partial z^2}\right)^2 \overline{\Psi}_{z(\zeta_{0i},z)} = -\overline{X}_{z(\zeta_{0i},z)} \tag{4.57}$$

where $\overline{\Psi}_{z(\zeta_{0i},z)}$ and $\overline{X}_{z(\zeta_{0i},z)}$ are the finite Hankel transform of the axial stress function and axial body force given by Eqs. (4.58) and (4.59), respectively.

$$\overline{\Psi}_{z(\zeta_{0i},z)} = \int_a^b r\Psi_{z(r,z)} K_{0(\zeta_{0i},r)} dr \tag{4.58}$$

$$\overline{X}_{z(\zeta_{0i},z)} = \int_a^b rX_{z(r,z)} K_{0(\zeta_{0i},r)} dr \tag{4.59}$$

Here, $K_{0(\zeta_{0i},r)}$ and ζ_{0i} correspond to zero order transformation.

$$K_{0(\zeta_{0i},r)} = \left[J_{0(\zeta_{0i}r)} Y_{0(\zeta_{0i}b)} - J_{0(\zeta_{0i}b)} Y_{0(\zeta_{0i}r)}\right] \tag{4.60}$$

$$J_{0(\zeta_{0i}a)} Y_{0(\zeta_{0i}b)} - J_{0(\zeta_{0i}b)} Y_{0(\zeta_{0i}a)} = 0 \tag{4.61}$$

By taking the finite Fourier sine transform

$$\Im_S\left[\left(-\zeta_{0i}^2+\frac{\partial^2}{\partial z^2}\right)^2\overline{\Psi}_{z(\zeta_{0i},z)}\right]=-\Im_S\left[\overline{X}_{z(\zeta_{0i},z)}\right] \tag{4.62}$$

the ordinary differential equation, Eq. (4.57), is converted into an algebraic equation expressed by

$$\left(-\zeta_{0i}^2-\frac{n^2\pi^2}{L^2}\right)^2\overline{\overline{\Psi}}_{z(\zeta_{0i},n)}=-\overline{\overline{X}}_{z(\zeta_{0i},n)} \tag{4.63}$$

where n is an integer number, and $\overline{\overline{\Psi}}_{z(\zeta_{0i},n)}$ and $\overline{\overline{X}}_{z(\zeta_{0i},n)}$ are the finite Fourier sine transforms of the $\overline{\Psi}_{z(\zeta_{0i},z)}$ and $\overline{X}_{z(\zeta_{0i},z)}$.

$$\overline{\overline{\Psi}}_{z(\zeta_{0i},n)}=\int_{-L}^{L}\overline{\Psi}_{z(\zeta_{0i},z)}\sin\frac{n\pi z}{L}dz \tag{4.64}$$

$$\overline{\overline{X}}_{z(\zeta_{0i},n)}=\int_{-L}^{L}\overline{X}_{z(\zeta_{0i},z)}\sin\frac{n\pi z}{L}dz \tag{4.65}$$

From Eq. (4.63) we can write $\overline{\overline{\Psi}}_{z(\zeta_{0i},n)}$ in terms of $\overline{\overline{X}}_{z(\zeta_{0i},n)}$.

$$\overline{\overline{\Psi}}_{z(\zeta_{0i},n)}=\frac{-L^4}{\left(L^2\zeta_{0i}^2+n^2\pi^2\right)^2}\overline{\overline{X}}_{z(\zeta_{0i},n)} \tag{4.66}$$

The inverse finite Fourier sine transform and finite Hankel transform are defined by Eqs. (4.67) and (4.68), respectively.

$$\overline{\Psi}_{z(\zeta_{0i},z)}=\Im_S^{-1}\left[\overline{\overline{\Psi}}_{r(\zeta_{0i},n)}\right]=\frac{1}{L}\sum_{n=1}^{\infty}\overline{\overline{\Psi}}_{z(\zeta_{0i},n)}\sin\frac{n\pi z}{L} \tag{4.67}$$

$$\Psi_{z(r,z)}=\Re_0^{-1}\left[\overline{\Psi}_{z(\zeta_{0i},z)}\right]=\frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\zeta_{0i}^2J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\overline{\Psi}_{z(\zeta_{0i},z)}K_0(\zeta_{0i},r) \tag{4.68}$$

Incorporating Eqs. (4.66) and (4.67) into Eq. (4.68) yields an expression for the axial stress function.

$$\Psi_{z(r,z)} = -\frac{\pi^2}{2}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{L^3}{\left(L^2\zeta_{0i}^2+n^2\pi^2\right)^2}\right. \qquad (4.69)$$

$$\left.\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\overline{\overline{X}}_{r(\zeta_{0i},n)}\sin\frac{n\pi z}{L}K_{0(\zeta_{0i},r)}\right]$$

Introducing Eqs. (4.59) and (4.65) into Eq. (4.69) gives the final solution to the axial stress function $\Psi_{z(r,z)}$ in terms of the axial body force $X_{r(r,z)}$.

$$\Psi_{z(r,z)} = -\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\pi^2}{2}\frac{L^3}{\left(L^2\zeta_{0i}^2+n^2\pi^2\right)^2}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\right. \qquad (4.70)$$

$$\left.\int_{-L}^{L}\int_{a}^{b}\left[r'K_{0(\zeta_{0i},r')}\sin\frac{n\pi z'}{L}X_{z(r',z')}dr'dz'\right]K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right\}$$

By interchanging integrals with summations, Eq. (4.70) may be written in the form

$$\Psi_{z(r,z)} = -\int_{-L}^{L}\int_{a}^{b}\left\{\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\pi^2}{2}\frac{L^3}{\left(L^2\zeta_{0i}^2+n^2\pi^2\right)^2}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\right.\right. \qquad (4.71)$$

$$\left.\left.r'K_{0(\zeta_{0i},r')}K_{0(\zeta_{0i},r)}\sin\frac{n\pi z'}{L}\sin\frac{n\pi z}{L}\right]\right\}X_{z(r',z')}dr'dz'$$

or

$$\Psi_{z(r,z)} = \int_{-L}^{L}\int_{a}^{b}X_{z(r',z')}\mathrm{G}_{z(r,r',z,z')}dr'dz' \qquad (4.72)$$

where $G_{z(r,r',z,z')}$ is the axial Green's function defined by Eq. (4.73).

$$G_{z(r,r',z,z')} = -\frac{\pi^2}{2}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{L^3}{\left(L^2\zeta_{0i}^2+n^2\pi^2\right)^2}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\right. \qquad (4.73)$$

$$\left.r'K_{0(\zeta_{0i},r')}K_{0(\zeta_{0i},r)}\sin\frac{n\pi z'}{L}\sin\frac{n\pi z}{L}\right]$$

Boundary Conditions

For traction free boundary conditions, the radial and shear stresses are zero at the inside radius, $r = a$, and outside radius, $r = b$, and the axial and shear stresses are zero at the ends of the coil, $z = \pm L$. Since the traction free boundary conditions were not considered for the solution of the stress functions, the expression for the stresses obtained from $\Psi_{r(r,z)}$ and $\Psi_{z(r,z)}$ will not satisfy these boundary conditions. Thus, radial and shear stresses impose new non zero forcing functions of z at radial boundaries, and similarly axial and shear stresses assert new forcing functions of r (in general non zero) at axial boundaries. To alleviate this apparent difficulty, other solutions (complementary solutions) for stress functions must be obtained in order to reverse the effect of these new imposed conditions. Since the partial differential equations for stress functions (Eqs. (4.13) and (4.15)) are linear, the superposition principle is applicable. Hence, the combination of the complementary solutions and the Green's functions furnish the final solutions for the stress functions. In this section the imposed forcing functions are derived, and in the succeeding section the complementary solutions are developed.

The displacement vector is related to the vector Ψ from Eqs. (4.6) and (4.8).

$$\mathbf{u} = \alpha\nabla \times (\nabla \times \Psi) + \beta\nabla(\nabla.\Psi) \tag{4.74}$$

The curl of the curl of any vector may be expressed by the gradient of the divergence of that vector minus the Laplacian.

$$\nabla \times (\nabla \times \Psi) = \nabla\left(\nabla.\Psi\right) - \nabla^2\Psi \tag{4.75}$$

Incorporating Eq. (4.75) into Eq. (4.74) and substituting Ψ by $\Psi_{r(r,z)}\mathbf{e_r} + \Psi_{z(r,z)}\mathbf{e_z}$ into Eq. (4.74), yields an equation for the displacement vector in terms of the stress functions, $\Psi_{r(r,z)}$ and $\Psi_{z(r,z)}$.

$$\mathbf{u} = -\alpha\nabla^2(\Psi_{r(r,z)}\mathbf{e_r} + \Psi_{z(r,z)}\mathbf{e_z}) + (\alpha+\beta)\nabla[\nabla.(\Psi_{r(r,z)}\mathbf{e_r} + \Psi_{z(r,z)}\mathbf{e_z}] \tag{4.76}$$

Rewriting Eq. (4.76) in component form results in expressions for radial and axial displacements, where φ is the divergence of vector Ψ given by Eq. (4.78).

$$u_r = -\alpha\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + (\alpha+\beta)\frac{\partial\varphi}{\partial r} \tag{4.77}$$

$$u_z = -\alpha\nabla^2\Psi_z + (\alpha+\beta)\frac{\partial\varphi}{\partial z}$$

$$\varphi = \frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r) + \frac{\partial\Psi_z}{\partial z} \tag{4.78}$$

The stress tensor in terms of the displacement vector is defined by

$$\boldsymbol{\sigma} = \lambda(\nabla.\mathbf{u})\mathbf{I} + \mu\left[\nabla\mathbf{u} + \nabla^{\mathrm{T}}\mathbf{u}\right].$$

In the component form it becomes

$$\sigma_r = \lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] + 2\mu\frac{\partial u_r}{\partial r} \tag{4.79}$$

$$\sigma_\theta = \lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] + 2\mu\frac{u_r}{r}$$

$$\sigma_z = \lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] + 2\mu\frac{\partial u_z}{\partial z}$$

$$\sigma_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).$$

The substitution of displacements from Eq. (4.77) into radial stress from Eq. (4.79), yields

$$\sigma_r = -\lambda\,\alpha\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\right] + \lambda(\alpha+\beta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) \tag{4.80}$$

$$-\lambda\,\alpha\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right) + \lambda(\alpha+\beta)\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right) - 2\mu\alpha\frac{\partial}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r$$

$$+2\mu(\alpha+\beta)\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right).$$

The terms $\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\right]$ and $\frac{\partial}{\partial z}(\nabla^2 \Psi_z)$ may be written as $\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r)\right]$ and $\nabla^2\left(\frac{\partial}{\partial z}\Psi_z\right)$, respectively. Hence, substitution of these terms into Eq. (4.80) gives

$$\sigma_r = -\lambda\alpha\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r)+\left(\frac{\partial \Psi_z}{\partial z}\right)\right]+\lambda(\alpha+\beta)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)+\frac{\partial^2\varphi}{\partial z^2}\right] \tag{4.81}$$
$$-2\mu\alpha\frac{\partial}{\partial r}\left(\nabla^2-\frac{1}{r^2}\right)\Psi_r+2\mu(\alpha+\beta)\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right),$$

but $\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r)+\frac{\partial \Psi_z}{\partial z}$ is φ and $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{\partial^2\varphi}{\partial z^2}$ is the Laplacian of φ. Thus, Eq. (4.81) becomes

$$\sigma_r = \lambda\beta\nabla^2\varphi-2\mu\alpha\frac{\partial}{\partial r}\left(\nabla^2-\frac{1}{r^2}\right)\Psi_r+2\mu(\alpha+\beta)\frac{\partial^2\varphi}{\partial r^2}. \tag{4.82}$$

The substitution of $\alpha = -\frac{1}{\mu}$ and $\beta = \frac{1}{\lambda+2\mu}$ in Eq. (4.82), yields

$$\sigma_r = \frac{\lambda}{\lambda+2\mu}\nabla^2\varphi+2\frac{\partial}{\partial r}\left(\nabla^2-\frac{1}{r^2}\right)\Psi_r+2(-1+\frac{\mu}{\lambda+2\mu})\frac{\partial^2\varphi}{\partial r^2}. \tag{4.83}$$

Lamé's elastic coefficients, λ and μ, are related to each other by

$$\lambda = \frac{2\mu v}{1-2v} \tag{4.84}$$

where v is the Poisson's ratio. Introducing Eq. (4.84) into Eq. (4.83) gives the radial stress in terms of stress functions.

$$\sigma_r = 2\frac{\partial}{\partial r}\left(\nabla^2-\frac{1}{r^2}\right)\Psi_r+\frac{1}{1-v}\left(v\nabla^2-\frac{\partial^2}{\partial r^2}\right)\varphi \tag{4.85}$$

In a similar fashion,* other components of stress are determined in terms of stress functions.

$$\sigma_\theta = \frac{2}{r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + \frac{1}{1-\nu}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\varphi \tag{4.86}$$

$$\sigma_z = 2\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right) + \frac{1}{1-\nu}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\varphi$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left(\nabla^2\Psi_z\right) + \frac{\partial}{\partial z}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r - \frac{1}{1-\nu}\frac{\partial^2\varphi}{\partial r\partial z}$$

Stress functions $\Psi_{r(r,z)}$ and $\Psi_{z(r,z)}$ are defined by $\int_{-L}^{L}\int_a^b X_{r(r',z')}G_{r(r,r',z,z')}dr'dz'$ and $\int_{-L}^{L}\int_a^b X_{z(r',z')}G_{z(r,r',z,z')}dr'dz'$ where $G_{r(r,r',z,z')}$ and $G_{z(r,r',z,z')}$ are radial and axial Green's functions given by Eqs. (4.55) and (4.73), respectively. Based upon Eqs. (4.55) and (4.73) and by introducing new functions $\Gamma_{r(\zeta_{1i},n)}$ and $\Gamma_{z(\zeta_{0i},n)}$, the stress functions may be written as

$$\Psi_{r(r,z)} = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}K_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L} \tag{4.87}$$

$$\Psi_{z(r,z)} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\Gamma_{z(\zeta_{0i},n)}\,K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L} \tag{4.88}$$

where $\Gamma_{r(\zeta_{1i},n)}$ and $\Gamma_{z(\zeta_{0i},n)}$ are given by Eqs. (4.89) and (4.90).

$$\Gamma_{z(\zeta_{0i},n)} = -\frac{\pi^2}{2}\left\{\frac{L^3}{\left(L^2\zeta_{0i}^2 + n^2\pi^2\right)^2}\frac{\zeta_{0i}^2 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\right. \tag{4.89}$$

$$\left.\int_{-L}^{L}\int_a^b r'X_{z(r',z')}\,K_{0(\zeta_{0i},r')}\sin\frac{n\pi z'}{L}dr'dz'\right\}$$

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* See Appendix D for details.

$$\Gamma_{r(\zeta_{1i},n)} = -\left\{\frac{\pi^2}{2}\frac{\zeta_{1i}^2 J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)}\frac{L^3}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2}\right. \tag{4.90}$$

$$\left.\int_{-L}^{L}\int_a^b r' X_{r(r',z')} K_{1(\zeta_{1i},r')} \cos\frac{n\pi z'}{L} dr'dz'\right\}$$

$$\Gamma_{r(\zeta_{1i},0)} = -\frac{\pi^2}{2}\frac{J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)}\frac{1}{2L\zeta_{1i}^2}\int_{-L}^{L}\int_a^b r' X_{r(r',z')} K_{1(\zeta_{1i},r')} dr'dz'$$

The substitution of Eqs. (4.88) and (4.87) into Eq. (4.85) gives

$$\sigma_r = 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{\partial}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\left[K_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right] \tag{4.91}$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\frac{1}{r}\frac{\partial}{\partial r}\left[rK_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\frac{\partial}{\partial z}\left[K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right].$$

The Fourier Bessel kernel $K_{1(\zeta_{1i},r)}$ is obtained from Bessel functions of order one. Hence, it satisfies the Bessel's equation which translates to

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\right]K_{1(\zeta_{1i},r)} = -\zeta_{1i}^2 K_{1(\zeta_{1i},r)}. \tag{4.92}$$

Also from basic identities for Bessel functions, we obtain

$$J'_{n(\zeta_{1i}r)} = J_{n-1(\zeta_{1i}r)} - \frac{n}{\zeta_{1i}r}J_{n(\zeta_{1i}r)} \tag{4.93}$$

$$Y'_{n(\zeta_{1i}r)} = Y_{n-1(\zeta_{1i}r)} - \frac{n}{\zeta_{1i}r}Y_{n(\zeta_{1i}r)}$$

or in terms of $K_{1(\zeta_{1i},r)}$

$$\frac{\partial}{\partial r}K_{1(\zeta_{1i},r)} = \zeta_{1i}\left[J_{0(\zeta_{1i}r)}Y_{1(\zeta_{1i}b)} - J_{0(\zeta_{1i}b)}Y_{1(\zeta_{1i}r)}\right] - \frac{1}{r}K_{1(\zeta_{1i},r)}. \tag{4.94}$$

Likewise for $K_0(\zeta_{0i},r)$ we may write

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]K_0(\zeta_{0i},r) = -\zeta_{0i}^2 K_0(\zeta_{0i},r) \tag{4.95}$$

$$\frac{\partial}{\partial r}K_0(\zeta_{0i},r) = -\zeta_{0i}\left[J_1(\zeta_{0i}r)Y_0(\zeta_{0i}b) - J_0(\zeta_{0i}b)Y_1(\zeta_{0i}r)\right].$$

Using results of Eqs. (4.92), (4.94) and (4.95) and by introducing new kernels $K_{01}(\zeta_{1i},r)$ and $K_{10}(\zeta_{0i},r)$, we can reduce Eq. (4.91) to

$$\sigma_r = 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_r(\zeta_{1i},n)\left[\left(-\zeta_{1i}^2 - \frac{n^2\pi^2}{L^2}\right)\left(\zeta_{1i}K_{01}(\zeta_{1i},r) - \frac{1}{r}K_1(\zeta_{1i},r)\right)\cos\frac{n\pi z}{L}\right] \tag{4.96}$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_r(\zeta_{1i},n)\left[\nu\left(-\zeta_{1i}^2 - \frac{n^2\pi^2}{L^2}\right)\zeta_{1i}K_{01}(\zeta_{1i},r)\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_r(\zeta_{1i},n)\left[\zeta_{1i}^2\left(\zeta_{1i}K_{01}(\zeta_{1i},r) - \frac{1}{r}K_1(\zeta_{1i},r)\right)\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_z(\zeta_{0i},n)\left(\frac{n\pi}{L}\right)\left[\nu\left(-\zeta_{0i}^2 - \frac{n^2\pi^2}{L^2}\right)K_0(\zeta_{0i},r)\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_z(\zeta_{0i},n)\left(\frac{n\pi}{L}\right)\left[\zeta_{0i}\left(\zeta_{0i}K_0(\zeta_{0i},r) - \frac{1}{r}K_{10}(\zeta_{0i},r)\right)\cos\frac{n\pi z}{L}\right]$$

where $K_{01}(\zeta_{1i},r)$ and $K_{10}(\zeta_{0i},r)$ are given by Eq. (4.97) and (4.98).

$$K_{01}(\zeta_{1i},r) = \left[J_0(\zeta_{1i}r)Y_1(\zeta_{1i}b) - J_1(\zeta_{1i}b)Y_0(\zeta_{1i}r)\right] = \frac{1}{r\zeta_{1i}}K_1(\zeta_{1i},r) + \frac{1}{\zeta_{1i}}\frac{\partial}{\partial r}K_1(\zeta_{1i},r) \tag{4.97}$$

$$K_{10}(\zeta_{0i},r) = \left[J_1(\zeta_{0i}r)Y_0(\zeta_{0i}b) - J_0(\zeta_{0i}b)Y_1(\zeta_{0i}r)\right] = -\frac{1}{\zeta_{0i}}\frac{\partial}{\partial r}K_0(\zeta_{0i},r) \tag{4.98}$$

Eq. (4.96) may be simplified further by combining the summations and canceling the common terms.

$$\sigma_r = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{r}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]rK_{0(\zeta_{0i},r)}\right. \tag{4.99}$$

$$-\Gamma_{z(\zeta_{0i},n)}\zeta_{0i}\frac{n\pi}{L}K_{10(\zeta_{0i},r)}$$

$$-\Gamma_{r(\zeta_{1i},n)}\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]rK_{01(\zeta_{1i},r)}$$

$$\left.+\Gamma_{r(\zeta_{1i},n)}\left[2\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{1i}^2(1-2\nu)\right]K_{1(\zeta_{1i},r)}\right\}\cos\frac{n\pi z}{L}$$

Likewise* we may write the other components of stress directly in terms of the Fourier Bessel kernels and trigonometric functions.

$$\sigma_\theta = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{r}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\nu\frac{n\pi}{L}\left(\frac{n^2\pi^2}{L^2}+\zeta_{0i}^2\right)rK_{0(\zeta_{0i},r)}\right. \tag{4.100}$$

$$+\Gamma_{z(\zeta_{0i},n)}\zeta_{0i}\frac{n\pi}{L}K_{10(\zeta_{0i},r)}$$

$$-\Gamma_{r(\zeta_{1i},n)}\nu\zeta_{1i}\left(\frac{n^2\pi^2}{L^2}+\zeta_{1i}^2\right)rK_{01(\zeta_{1i},r)}$$

$$\left.-\Gamma_{r(\zeta_{1i},n)}\left[2\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{1i}^2(1-2\nu)\right]K_{1(\zeta_{1i},r)}\right\}\cos\frac{n\pi z}{L}$$

$$\sigma_z = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\left[\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{0i}^2(2-\nu)\right]K_{0(\zeta_{0i},r)}\right. \tag{4.101}$$

$$\left.+\Gamma_{r(\zeta_{1i},n)}\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(1-\nu)-\nu\zeta_{1i}^2\right]K_{01(\zeta_{1i},r)}\right\}\cos\frac{n\pi z}{L}$$

$$\sigma_{rz} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\zeta_{0i}\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]K_{10(\zeta_{0i},r)}\right. \tag{4.102}$$

$$\left.+\frac{n\pi}{L}\Gamma_{r(\zeta_{1i},n)}\left[\frac{n^2\pi^2}{L^2}(1-\nu)-\nu\zeta_{1i}^2\right]K_{1(\zeta_{1i},r)}\right\}\sin\frac{n\pi z}{L}$$

The above expressions for stresses satisfy the field equation but not the boundary

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* See Appendix E for details.

conditions. At the inner radius, by the use of $K_{1}(\zeta_{1i}, a) = K_{0}(\zeta_{0i}, a) = 0$, and from Eqs. (4.103) and (4.104), the radial and shear stresses are determined.

$$\sigma_{r}(a,z) = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{1-\nu}\left\{-\Gamma_{z}(\zeta_{0i},n)\zeta_{0i}\frac{n\pi}{L}\frac{1}{a}K_{10}(\zeta_{0i},a)\right.$$
$$\left.-\Gamma_{r}(\zeta_{1i},n)\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]K_{01}(\zeta_{1i},a)\right\}\cos\frac{n\pi z}{L} \tag{4.103}$$

$$\sigma_{rz}(a,z) = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left\{-\frac{1}{1-\nu}\Gamma_{z}(\zeta_{0i},n)\zeta_{0i}\right.$$
$$\left.\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]K_{10}(\zeta_{0i},a)\sin\frac{n\pi z}{L}\right\} \tag{4.104}$$

From Eqs. (4.93) and (4.97) we may write

$$K_{01}(\zeta_{1i},a) = \frac{1}{a\zeta_{1i}}K_{1}(\zeta_{1i},a) + \frac{1}{\zeta_{1i}}\frac{\partial}{\partial r}K_{1}(\zeta_{1i},r)\bigg|_{r=a}$$

and

$$K_{10}(\zeta_{0i},a) = -\frac{1}{\zeta_{0i}}\left.\frac{\partial K_{0}(\zeta_{0i},r)}{\partial r}\right|_{r=a}.$$

However, $K_{1}(\zeta_{1i},a)$ is zero and $\left.\frac{\partial}{\partial r}K_{1}(\zeta_{1i},r)\right|_{r=a}$ and $\left.\frac{\partial}{\partial r}K_{0}(\zeta_{0i},r)\right|_{r=a}$ may be expressed* by $-\frac{2}{\pi a}\frac{J_{1}(\zeta_{1i}b)}{J_{1}(\zeta_{1i}a)}$ and $-\frac{2}{\pi a}\frac{J_{0}(\zeta_{0i}b)}{J_{0}(\zeta_{0i}a)}$, respectively. Thus, Eqs. (4.103) and (4.104) are reduced to

$$\sigma_{r}(a,z) = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{2}{\pi a^2}\frac{1}{1-\nu}\left\{-\Gamma_{z}(\zeta_{0i},n)\frac{n\pi}{L}\frac{J_{0}(\zeta_{0i}b)}{J_{0}(\zeta_{0i}a)}\right.$$
$$\left.+a\frac{J_{1}(\zeta_{1i}b)}{J_{1}(\zeta_{1i}a)}\Gamma_{r}(\zeta_{1i},n)\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]\right\}\cos\frac{n\pi z}{L} \tag{4.105}$$

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* See Eq. (C.37) of Appendix C.

$$\sigma_{rz(a,z)} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left\{-\Gamma_{z(\zeta_{0i},n)}\left[\nu\frac{n^2\pi^2}{L^2} - \zeta_{0i}^2(1-\nu)\right]\right. \tag{4.106}$$

$$\left.\frac{J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}}\frac{2}{\pi a}\frac{1}{1-\nu}\sin\frac{n\pi z}{L}\right\}.$$

By introducing new functions $\wp_{1(n)}$ and $\wp_{2(n)}$ as

$$\wp_{1(n)} = \sum_{i=1}^{\infty}\frac{2}{\pi a^2}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\frac{J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}}\right. \tag{4.107}$$

$$\left.+a\Gamma_{r(\zeta_{1i},n)}\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]\frac{J_{1(\zeta_{1i}b)}}{J_{1(\zeta_{1i}a)}}\right\}$$

$$\wp_{2(n)} = \sum_{i=1}^{\infty}\frac{2}{\pi a}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\left[\nu\frac{n^2\pi^2}{L^2} - \zeta_{0i}^2(1-\nu)\right]\frac{J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}}\right\}, \tag{4.108}$$

we may simplify Eqs. (4.105) and (4.106) to Eqs. (4.109) and (4.110), respectively.

$$\sigma_{r(a,z)} = \sum_{n=0}^{\infty}\wp_{1(n)}\cos\frac{n\pi z}{L} \tag{4.109}$$

$$\sigma_{rz(a,z)} = \sum_{n=1}^{\infty}\wp_{2(n)}\sin\frac{n\pi z}{L} \tag{4.110}$$

At outer radius, $r = b$, with use of the expressions*

$$\left.\frac{\partial}{\partial r}K_{1(\zeta_{1i},r)}\right|_{r=b} = -\frac{2}{\pi b}$$

and

$$\left.\frac{\partial}{\partial r}K_{0(\zeta_{0i},r)}\right|_{r=b} = -\frac{2}{\pi b},$$

Eqs. (4.105) and (4.106) may be written as

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* See Eq. (C.37) of Appendix C.

$$\sigma_{r(b,z)} = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{2}{\pi b^2}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\right. \tag{4.111}$$

$$\left. +b\Gamma_{r(\zeta_{1i},n)}\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]\right\}\cos\frac{n\pi z}{L}$$

$$\sigma_{rz(b,z)} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{1-\nu}\frac{2}{\pi b}\left\{-\Gamma_{z(\zeta_{0i},n)}\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]\right\}\sin\frac{n\pi z}{L} \tag{4.112}$$

or

$$\sigma_{r(b,z)} = \sum_{n=0}^{\infty}\wp_{3(n)}\cos\frac{n\pi z}{L} \tag{4.113}$$

$$\sigma_{rz(b,z)} = \sum_{n=1}^{\infty}\wp_{4(n)}\sin\frac{n\pi z}{L} \tag{4.114}$$

where $\wp_{3(n)}$ and $\wp_{4(n)}$ are given by Eqs. (4.115) and (4.116).

$$\wp_{3(n)} = \sum_{i=1}^{\infty}\frac{2}{\pi b^2}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}+b\Gamma_{r(\zeta_{1i},n)}\left[\frac{n^2\pi^2}{L^2}(2-\nu)+\zeta_{1i}^2(1-\nu)\right]\right\} \tag{4.115}$$

$$\wp_{4(n)} = \sum_{i=1}^{\infty}\frac{1}{1-\nu}\frac{2}{\pi b}\left\{-\Gamma_{z(\zeta_{0i},n)}\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]\right\} \tag{4.116}$$

Since $\sin\frac{n\pi z}{L}$ vanishes at both L and $-L$, shear stress at the axial boundaries does not exert a forcing function. However, axial stress is a function of cosine and does not vanish at boundaries. The substitution of $z = \pm L$ into axial stress, Eq. (4.101), yields

$$\sigma_{z(r,\pm L)} = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\left[\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{0i}^2(2-\nu)\right]K_{0(\zeta_{0i},r)}\right. \tag{4.117}$$

$$\left. +\Gamma_{r(\zeta_{1i},n)}\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(1-\nu)-\nu\zeta_{1i}^2\right]K_{01(\zeta_{1i},r)}\right\}$$

or

$$\sigma_{z(r,\pm L)} = \sum_{i=1}^{\infty}\left[\wp_{5(\zeta_{0i})}K_{0(\zeta_{0i},r)} + \wp_{6(\zeta_{1i})}K_{01(\zeta_{1i},r)}\right] \tag{4.118}$$

where $\wp_{5(\zeta_{0i})}$ and $\wp_{6(\zeta_{1i})}$ are given by (4.119) and (4.120).

$$\wp_{5(\zeta_{0i})} = \sum_{n=0}^{\infty}\frac{(-1)^n}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\left[\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{0i}^2(2-\nu)\right]\right\} \tag{4.119}$$

$$\wp_{6(\zeta_{1i})} = \sum_{n=0}^{\infty}\frac{(-1)^n}{1-\nu}\left\{\Gamma_{r(\zeta_{1i},n)}\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(1-\nu)-\nu\zeta_{1i}^2\right]\right\} \tag{4.120}$$

Eqs. (4.109), (4.110), (4.113) and (4.114) together with Eq. (4.118) dictate the forcing functions at the boundaries. Complementary solutions for stress functions must be obtained that reverse the effect of these forcing functions. Looking at the equations for stresses, we observe that all of the components of stress are related to both radial and axial stress functions. Hence, either one of the stress functions may be used to reverse the effect of the forcing functions. Here we use the axial stress function. Because the original partial differential equation is satisfied by the axial Green's function solution, we require that the complementary solution only satisfies the homogeneous part of the partial differential equation.

Complementary Solution for the Axial Stress Function

Consider function $\xi_{(r,z)}$ as a complementary solution to the axial stress function. From Eq. (4.15), $\xi_{(r,z)}$ satisfies the homogeneous partial differential equation of the axial stress function.

$$\nabla^4\xi_{(r,z)} = 0 \tag{4.121}$$

From Eqs. (4.78), (4.85), and (4.86), radial, axial, and shear stresses may be written in terms of $\xi_{(r,z)}$.

$$\sigma_r = \frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\xi_{(r,z)} \tag{4.122}$$

$$\sigma_z = \frac{1}{1-\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\xi_{(r,z)} \tag{4.123}$$

$$\sigma_{rz} = \frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\xi_{(r,z)} \tag{4.124}$$

The function $\xi_{(r,z)}$ must reverse the effect of the imposed forcing functions by stresses at the boundaries. Thus, from Eqs. (4.109), (4.110), (4.113), (4.114), (4.118), (4.122), (4.123) and (4.124) boundary conditions for $\xi_{(r,z)}$ may be expressed by

$$\left.\frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\xi_{(r,z)}\right|_{r=a} = -\sum_{n=0}^{\infty}\wp_{1(n)}\cos\frac{n\pi z}{L} \tag{4.125}$$

$$\left.\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\xi_{(r,z)}\right|_{r=a} = -\sum_{n=1}^{\infty}\wp_{2(n)}\sin\frac{n\pi z}{L} \tag{4.126}$$

$$\left.\frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\xi_{(r,z)}\right|_{r=b} = -\sum_{n=0}^{\infty}\wp_{3(n)}\cos\frac{n\pi z}{L} \tag{4.127}$$

$$\left.\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\xi_{(r,z)}\right|_{r=b} = -\sum_{n=1}^{\infty}\wp_{4(n)}\sin\frac{n\pi z}{L} \tag{4.128}$$

$$\left.\frac{1}{1-\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\xi_{(r,z)}\right|_{z=\pm L} = -\sum_{i=1}^{\infty}\left[\wp_{5(\zeta_{0i})}K_{0(\zeta_{0i},r)} + \wp_{6(\zeta_{1i})}K_{01(\zeta_{1i},r)}\right] \tag{4.129}$$

where $\wp_{1(n)}$, $\wp_{2(n)}$, $\wp_{3(n)}$, $\wp_{4(n)}$, $\wp_{5(\zeta_{0i})}$ and $\wp_{6(\zeta_{0i})}$ are given by Eqs. (4.107), (4.108), (4.115), (4.116), (4.119) and (4.120), respectively. Eqs. (4.125) through (4.129) specify six boundary conditions for $\xi_{(r,z)}$, however, eight boundary conditions are required for solving Eq. (4.121). Hence, the two needed boundary conditions are derived from the fact that the shear stress does not exert a forcing function at two ends of the coil. From Eq. (4.124), we may write

$$\frac{1}{1-\nu}\frac{\partial}{\partial r}\left((1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right)\xi_{(r,z)}\bigg|_{z=\pm L}=0. \tag{4.130}$$

The substitution of $\xi_{(r,z)}=\xi_{1(r,z)}+\xi_{2(r,z)}$ into equation (4.121) yields two partial differential equations for $\xi_{1(r,z)}$ and $\xi_{2(r,z)}$.

$$\nabla^4\xi_{1(r,z)}=0 \tag{4.131}$$

$$\nabla^4\xi_{2(r,z)}=0 \tag{4.132}$$

The nonhomogeneous boundary conditions for $\xi_{(r,z)}$ do not fall into the four traditional (Cauchy, Dirichlet, Neumann and Robin) categories. Thus, the method of solving the differential equation for $\xi_{(r,z)}$ is somehow different from the usual transient method. In the usual transient method, the radial nonhomogeneous with the axial homogeneous boundary conditions and the axial nonhomogeneous with the radial homogeneous boundary conditions are assigned to $\xi_{1(r,z)}$ and $\xi_{2(r,z)}$, respectively. Thus, $\xi_{1(r,z)}$ and $\xi_{2(r,z)}$ each will have four nonhomogeneous boundary conditions in one direction and four homogeneous boundary conditions in the other direction. Then from homogeneous boundary conditions and using one of the conventional methods, e.g., eigenfunction expansion, the partial differential equation is reduced to an ordinary differential equation. The resulting arbitrary constants from the solution to this ordinary differential equation are then calculated from the nonhomogeneous boundary conditions. Hence, solutions for

$\xi_{1(r,z)}$ and $\xi_{2(r,z)}$ are obtained independently and the final solution for $\xi_{(r,z)}$ is determined from the superposition of these two solutions.

However, In the present formulation, equations for the boundary conditions are very complicated and as a result it is not possible to reduce the partial differential equations to ordinary differential equations by using the homogeneous form of these boundary conditions. Therefore, by assuming homogeneous Dirichlet boundary conditions in the radial and axial directions for $\xi_{1(r,z)}$ and $\xi_{2(r,z)}$, respectively, the partial differential equations are reduced to ordinary differential equations. Solutions to these ordinary differential equations are obtained without applying the nonhomogeneous boundary conditions and they are superposed to give the solution for $\xi_{(r,z)}$. Thus, the solution for $\xi_{(r,z)}$ will have eight arbitrary constants, which can be evaluated from the eight original boundary conditions, Eqs. (4.125)-(4.130).

The partial differential equation for $\xi_{1(r,z)}$ is assumed to have homogeneous Dirichlet boundary conditions in r. Thus, by using the finite Hankel transform (of order zero) in the radial direction, the partial differential equation for $\xi_{1(r,z)}$ is reduced to an ordinary differential equation. Applying the finite Hankel transform to Eq. (4.131) yields

$$\Re_0[\nabla^4 \xi_1] = \left(-\zeta_{0i}^2 + \frac{\partial^2}{\partial z^2}\right)^2 \overline{\xi}_{1(\zeta_{0i},z)} = 0 \tag{4.133}$$

where $\overline{\xi}_{1(\zeta_{0i},z)}$ is the transform of the $\xi_{1(r,z)}$.

$$\overline{\xi}_{1(\zeta_{0i},z)} = \int_a^b r\xi_{1(r,z)} K_{0(\zeta_{0i},r)} dr \tag{4.134}$$

Eq. (4.133) is a linear ordinary differential equation in z with constant coefficients. Solution to this differential equation may be expressed by

$$\overline{\xi}_{1(\zeta_{0i},z)} = Ae^{\zeta_{0i}z} + Be^{-\zeta_{0i}z} + Cze^{\zeta_{0i}z} + Dze^{-\zeta_{0i}z} \tag{4.135}$$

where A, B, C and D are arbitrary constants. By taking the inverse finite Hankel transform of Eq. (4.135), the solution for $\xi_{1(r,z)}$ is obtained.

$$\xi_{1(r,z)} = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}\left[A_i e^{\zeta_{0i}z} + B_i e^{-\zeta_{0i}z} + C_i z e^{\zeta_{0i}z} + D_i z e^{-\zeta_{0i}z}\right]K_0(\zeta_{0i}, r) \tag{4.136}$$

The partial differential equation for $\xi_{2(r,z)}$ has homogeneous Dirichlet boundary conditions in z. Thus, the finite Fourier sine transform in axial direction is used to reduce the partial differential equation for $\xi_{2(r,z)}$ to an ordinary differential equation. Hence, by applying the finite Fourier sine transform, we may write

$$\Im_s\left[\nabla^4 \xi_{2(r,z)}\right] = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right)^2 \bar{\xi}_{2(r,n)} = 0 \tag{4.137}$$

where $\bar{\xi}_{2(r,n)}$ is the finite Fourier sine transform of the function $\xi_{2(r,z)}$.

$$\bar{\xi}_{2(r,n)} = \int_{-L}^{L} \xi_{2(r,z)} \sin\frac{n\pi z}{L}\,dz \tag{4.138}$$

Eq. (4.137) is a linear ordinary differential equation in r. Solution to this equation may obtained by introducing a new function $\Xi_{(r,n)}$.

$$\Xi_{(r,n)} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right)\bar{\xi}_{2(r,n)} \tag{4.139}$$

Incorporating Eq. (4.139) to Eq. (4.137), yields a differential equation for $\Xi_{(r,n)}$.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right)\Xi_{(r,n)} = 0 \tag{4.140}$$

Eq. (4.140) is a modified Bessel equation of order zero. Solution to this equation is given by

$$\Xi_{(r,n)} = \hat{A} I_{0(\frac{n\pi}{L}r)} + \hat{B} K_{0(\frac{n\pi}{L}r)} \tag{4.141}$$

where $\hat{A}$ and $\hat{B}$ are arbitrary constants, and $I_{0(\frac{n\pi}{L}r)}$ and $K_{0(\frac{n\pi}{L}r)}$ are the modified Bessel functions of the first and second kind (order zero), respectively. The substitution of Eq. (4.141) into Eq. (4.139) gives a differential equation for $\overline{\xi}_{2(r,n)}$.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right)\overline{\xi}_{2(r,n)} = \hat{A} I_{0(\frac{n\pi}{L}r)} + \hat{B} K_{0(\frac{n\pi}{L}r)} \tag{4.142}$$

The homogenous and particular solutions to Eq. (4.142) may be written as

$$\overline{\xi}_{2(r,n)} = \hat{C} I_{0(\frac{n\pi}{L}r)} + \hat{D} K_{0(\frac{n\pi}{L}r)} + \hat{E} r I_{1(\frac{n\pi}{L}r)} + \hat{F} r K_{1(\frac{n\pi}{L}r)} \tag{4.143}$$

where $\hat{C}$ and $\hat{D}$ are arbitrary constants, and $\hat{E}$ and $\hat{F}$ are constants that must be determined from the nonhomogeneous part of Eq. (4.142). Substituting Eq. (4.143) into Eq.(4.142) results in

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right)\left(\hat{E} r I_{1(\frac{n\pi}{L}r)} + \hat{F} r K_{1(\frac{n\pi}{L}r)}\right) = \hat{A} I_{0(\frac{n\pi}{L}r)} + \hat{B} K_{0(\frac{n\pi}{L}r)}. \tag{4.144}$$

Expanding the left hand side of Eq. (4.144), leads to

$$\begin{aligned}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right)&\left(\hat{E} r I_{1(\frac{n\pi}{L}r)} + \hat{F} r K_{1(\frac{n\pi}{L}r)}\right) \\ &= \left[r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right) I_{1(\frac{n\pi}{L}r)} + \frac{1}{r} I_{1(\frac{n\pi}{L}r)} + 2\frac{\partial}{\partial r} I_{1(\frac{n\pi}{L}r)} \hat{E}\right] \\ &+ \left[r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2\pi^2}{L^2}\right) K_{1(\frac{n\pi}{L}r)} + \frac{1}{r} K_{1(\frac{n\pi}{L}r)} + 2\frac{\partial}{\partial r} K_{1(\frac{n\pi}{L}r)} \hat{F}\right].\end{aligned} \tag{4.145}$$

From the modified Bessel equation we may write

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2\pi^2}{L^2}\right)I_{1(\frac{n\pi}{L}r)}=\frac{1}{r^2}I_{1(\frac{n\pi}{L}r)}$$

and

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2\pi^2}{L^2}\right)K_{1(\frac{n\pi}{L}r)}=\frac{1}{r^2}K_{1(\frac{n\pi}{L}r)}.$$

Introducing these expressions into Eq. (4.145), yields

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2\pi^2}{L^2}\right)\left(\hat{E}rI_{1(\frac{n\pi}{L}r)}+\hat{F}rK_{1(\frac{n\pi}{L}r)}\right) \tag{4.146}$$

$$=\left[\frac{2}{r}I_{1(\frac{n\pi}{L}r)}+2\frac{\partial}{\partial r}I_{1(\frac{n\pi}{L}r)}\right]\hat{E}+\left[\frac{2}{r}K_{1(\frac{n\pi}{L}r)}+2\frac{\partial}{\partial r}K_{1(\frac{n\pi}{L}r)}\right]\hat{F}.$$

By using identities $I'_{1(\frac{n\pi}{L}r)}=I_{0(\frac{n\pi}{L}r)}-\frac{1}{\frac{n\pi}{L}r}I_{1(\frac{n\pi}{L}r)}$ and $K'_{1(\frac{n\pi}{L}r)}=K_{0(\frac{n\pi}{L}r)}-\frac{1}{\frac{n\pi}{L}r}K_{1(\frac{n\pi}{L}r)}$ Eq. (4.146) reduces to

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2\pi^2}{L^2}\right)\left(\hat{E}rI_{1(\frac{n\pi}{L}r)}+\hat{F}rK_{1(\frac{n\pi}{L}r)}\right)$$

$$=\left[\frac{2}{r}I_{1(\frac{n\pi}{L}r)}+2\left(\frac{n\pi}{L}I_{0(\frac{n\pi}{L}r)}-\frac{1}{r}I_{1(\frac{n\pi}{L}r)}\right)\right]\hat{E}$$

$$+\left[\frac{2}{r}K_{1(\frac{n\pi}{L}r)}+2\left(\frac{n\pi}{L}K_{0(\frac{n\pi}{L}r)}-\frac{1}{r}K_{1(\frac{n\pi}{L}r)}\right)\right]\hat{F}$$

or

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2\pi^2}{L^2}\right)\left(\hat{E}rI_{1(\frac{n\pi}{L}r)}+\hat{F}rK_{1(\frac{n\pi}{L}r)}\right) \tag{4.147}$$

$$=2\frac{n\pi}{L}\hat{E}I_{1(\frac{n\pi}{L}r)}+2\frac{n\pi}{L}\hat{F}K_{1(\frac{n\pi}{L}r)}.$$

Incorporating Eq. (4.147) into Eq. (4.144) gives expression for $\hat{E}$ and $\hat{F}$.

$$\hat{E} = \frac{L}{2n\pi}\hat{A} \tag{4.148}$$

$$\hat{F} = \frac{L}{2n\pi}\hat{B}$$

Hence, $\overline{\xi}_{2(r,n)}$ may be written as

$$\overline{\xi}_{2(r,n)} = \hat{C}I_{0(\frac{n\pi}{L}r)} + \hat{D}K_{0(\frac{n\pi}{L}r)} + \frac{L}{2n\pi}\hat{A}\ rI_{1(\frac{n\pi}{L}r)} + \frac{L}{2n\pi}\hat{B}rK_{1(\frac{n\pi}{L}r)}. \tag{4.149}$$

The solution to $\xi_{2(r,z)}$ is simply the inverse finite Fourier transform of $\overline{\xi}_{2(r,n)}$.

$$\xi_{2(r,z)} = \Im_S^{-1}\left[\overline{\xi}_{2(r,n)}\right] = \frac{1}{L}\sum_{n=1}^{\infty}\overline{\xi}_{2(r,n)}\sin\frac{n\pi z}{L} \tag{4.150}$$

Introducing Eq. (4.149) into Eq. (4.150), gives the final form for $\xi_{2(r,z)}$.

$$\xi_{2(r,z)} = \frac{1}{L}\sum_{n=1}^{\infty}\Big[\hat{C}_n I_{0(\frac{n\pi}{L}r)} + \hat{D}_n K_{0(\frac{n\pi}{L}r)} \tag{4.151}$$
$$+ \frac{L}{2n\pi}\hat{A}_n rI_{1(\frac{n\pi}{L}r)} + \frac{L}{2n\pi}\hat{B}_n rK_{1(\frac{n\pi}{L}r)}\Big]\sin\frac{n\pi z}{L}$$

The superposition of Eqs. (4.136) and (4.151) furnishes the solution to $\xi_{(r,z)}$.

$$\xi_{(r,z)} = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{\zeta_{0i}^2 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\Big[A_i e^{\zeta_{0i}z} + B_i e^{-\zeta_{0i}z} \tag{4.152}$$
$$+ C_i z e^{\zeta_{0i}z} + D_i z e^{-\zeta_{0i}z}\Big]K_{0(\zeta_{0i},r)}$$
$$+ \frac{1}{L}\sum_{n=1}^{\infty}\Big[\hat{C}_n I_{0(\frac{n\pi}{L}r)} + \hat{D}_n K_{0(\frac{n\pi}{L}r)}$$
$$+ \frac{L}{2n\pi}\hat{A}_n rI_{1(\frac{n\pi}{L}r)} + \frac{L}{2n\pi}\hat{B}_n rK_{1(\frac{n\pi}{L}r)}\Big]\sin\frac{n\pi z}{L}$$

Function $\xi_{(r,z)}$ is an odd function in z, hence $\xi_{(r,z)} = -\xi_{(r,-z)}$. Incorporating this expression into Eq. (4.152) leads to

$$\left[A_i e^{\zeta_{0i}z} + B_i e^{-\zeta_{0i}z} + C_i z e^{\zeta_{0i}z} + D_i z e^{-\zeta_{0i}z}\right] \tag{4.153}$$

$$= -\left[A_i e^{-\zeta_{0i}z} + B_i e^{\zeta_{0i}z} + C_i z e^{-\zeta_{0i}z} - D_i z e^{\zeta_{0i}z}\right]$$

or

$$A_i = -B_i \tag{4.154}$$

$$C_i = D_i.$$

Thus, the expression for $\xi_{(r,z)}$ reduces to

$$\xi_{(r,z)} = \pi^2 \sum_{i=1}^{\infty} \frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)} [A_i \sinh(\zeta_{0i}z) \tag{4.155}$$

$$+ C_i z \cosh(\zeta_{0i}z)] K_0(\zeta_{0i}, r)$$

$$+ \frac{1}{L}\sum_{n=1}^{\infty}\left[\hat{C}_n I_0(\tfrac{n\pi}{L}r) + \hat{D}_n K_0(\tfrac{n\pi}{L}r)\right.$$

$$\left. + \frac{L}{2n\pi}\hat{A}_n r I_1(\tfrac{n\pi}{L}r) + \frac{L}{2n\pi}\hat{B}_n r K_1(\tfrac{n\pi}{L}r)\right] \sin\frac{n\pi z}{L}$$

where $\sinh(\zeta_{0i}z)$ and $\cosh(\zeta_{0i}z)$ are hyperbolic sine and cosine, respectively.

Eqs. (4.125) through (4.130) describe the boundary conditions for $\xi_{(r,z)}$. In order to evaluate the boundary conditions, we need first to derive the following expressions.

Laplacian of $\xi_{(r,z)}$:

$$\nabla^2 \xi_{(r,z)} = \pi^2 \sum_{i=1}^{\infty} \frac{2\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)} C_i \sinh(\zeta_{0i}z) K_0(\zeta_{0i}, r) \tag{4.156}$$

$$+ \frac{1}{L}\sum_{n=1}^{\infty}\left[\hat{A}_n I_0(\tfrac{n\pi}{L}r) + \hat{B}_n K_0(\tfrac{n\pi}{L}r)\right] \sin\frac{n\pi z}{L}$$

Second partial derivatives of $\xi_{(r,z)}$ with respect to z:

$$\frac{\partial^2}{\partial z^2}\xi_{(r,z)} = \pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^4 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}[A_i\sinh(\zeta_{0i}z) \tag{4.157}$$

$$+ C_i z\cosh(\zeta_{0i}z)]K_0(\zeta_{0i},r)$$

$$+\pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}C_i\sinh(\zeta_{0i}z)K_0(\zeta_{0i},r)$$

$$-\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^3}\Big[\hat{C}_n I_0(\tfrac{n\pi}{L}r) + \hat{D}_n K_0(\tfrac{n\pi}{L}r)$$

$$+\frac{L}{2n\pi}\hat{A}_n r I_1(\tfrac{n\pi}{L}r) + \frac{L}{2n\pi}\hat{B}_n r K_1(\tfrac{n\pi}{L}r)\Big]\sin\frac{n\pi z}{L}$$

Second partial derivatives of $\xi_{(r,z)}$ with respect to r:

$$\frac{\partial^2}{\partial r^2}\xi_{(r,z)} = -\pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^4 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}[A_i\sinh(\zeta_{0i}z) \tag{4.158}$$

$$+ C_i z\cosh(\zeta_{0i}z)]K_0(\zeta_{0i},r)$$

$$-\pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}[A_i\sinh(\zeta_{0i}z)$$

$$+ C_i z\cosh(\zeta_{0i}z)]\frac{1}{r}\frac{\partial}{\partial r}K_0(\zeta_{0i},r)$$

$$+\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^3}\Big[\hat{C}_n I_0(\tfrac{n\pi}{L}r) + \hat{D}_n K_0(\tfrac{n\pi}{L}r)$$

$$+\frac{L}{2n\pi}\hat{A}_n r I_1(\tfrac{n\pi}{L}r) + \frac{L}{2n\pi}\hat{B}_n r K_1(\tfrac{n\pi}{L}r)\Big]\sin\frac{n\pi z}{L}$$

$$-\sum_{n=1}^{\infty}\frac{n\pi}{L^2}\frac{1}{r}\Big[\hat{C}_n I_1(\tfrac{n\pi}{L}r) + \hat{D}_n K_1(\tfrac{n\pi}{L}r)\Big]\sin\frac{n\pi z}{L}$$

$$+\frac{1}{2L}\sum_{n=1}^{\infty}\Big[\hat{A}_n I_0(\tfrac{n\pi}{L}r) + \hat{B}_n K_0(\tfrac{n\pi}{L}r)\Big]\sin\frac{n\pi z}{L}$$

First partial derivatives of $\nabla^2\xi_{(r,z)}$ with respect to z:

$$\frac{\partial}{\partial z}\nabla^2\xi_{(r,z)} = \pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^4 J_{0\,(\zeta_{0i}a)}^2}{J_{0\,(\zeta_{0i}a)}^2 - J_{0\,(\zeta_{0i}b)}^2} C_i \cosh(\zeta_{0i}z) K_{0\,(\zeta_{0i},r)}$$

$$+\sum_{n=1}^{\infty}\frac{n\pi}{L^2}[\hat{A}_n I_{0\,(\frac{n\pi}{L}r)}$$

$$+\hat{B}_n K_{0\,(\frac{n\pi}{L}r)}]\cos\frac{n\pi z}{L} \tag{4.159}$$

First partial derivatives of $\nabla^2\xi_{(r,z)}$ with respect to r:

$$\frac{\partial}{\partial r}\nabla^2\xi_{(r,z)} = \pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^3 J_{0\,(\zeta_{0i}a)}^2}{J_{0\,(\zeta_{0i}a)}^2 - J_{0\,(\zeta_{0i}b)}^2} C_i \sinh(\zeta_{0i}z)\frac{\partial}{\partial r} K_{0\,(\zeta_{0i},r)}$$

$$+\sum_{n=1}^{\infty}\frac{n\pi}{L^2}[\hat{A}_n I_{1\,(\frac{n\pi}{L}r)}$$

$$+\hat{B}_n K_{1\,(\frac{n\pi}{L}r)}]\sin\frac{n\pi z}{L} \tag{4.160}$$

Third partial derivatives of $\xi_{(r,z)}$ with respect to z:

$$\frac{\partial^3}{\partial z^3}\xi_{(r,z)} = \pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^5 J_{0\,(\zeta_{0i}a)}^2}{J_{0\,(\zeta_{0i}a)}^2 - J_{0\,(\zeta_{0i}b)}^2}[A_i \cosh(\zeta_{0i}z)$$

$$+C_i z\sinh(\zeta_{0i}z)]K_{0\,(\zeta_{0i},r)}$$

$$+\pi^2\sum_{i=1}^{\infty}\frac{3\zeta_{0i}^4 J_{0\,(\zeta_{0i}a)}^2}{J_{0\,(\zeta_{0i}a)}^2 - J_{0\,(\zeta_{0i}b)}^2} C_i \cosh(\zeta_{0i}z) K_{0\,(\zeta_{0i},r)}$$

$$-\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\Big[\hat{C}_n I_{0\,(\frac{n\pi}{L}r)} + \hat{D}_n K_{0\,(\frac{n\pi}{L}r)}$$

$$+\frac{L}{2n\pi}\hat{A}_n r I_{1\,(\frac{n\pi}{L}r)} + \frac{L}{2n\pi}\hat{B}_n r K_{1\,(\frac{n\pi}{L}r)}\Big]\cos\frac{n\pi z}{L} \tag{4.161}$$

First partial derivatives of $\xi_{(r,z)}$ with respect to r and second partial derivative with respect to z:

$$\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\xi_{(r,z)} = \pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^4 J_0^2{}_{(\zeta_{0i}a)}}{J_0^2{}_{(\zeta_{0i}a)} - J_0^2{}_{(\zeta_{0i}b)}}\big[A_i\sinh(\zeta_{0i}z) \tag{4.162}$$

$$+C_i z\cosh(\zeta_{0i}z)\big]\frac{\partial}{\partial r}K_{0(\zeta_{0i},r)}$$

$$+\pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^3 J_0^2{}_{(\zeta_{0i}a)}}{J_0^2{}_{(\zeta_{0i}a)} - J_0^2{}_{(\zeta_{0i}b)}}C_i\sinh(\zeta_{0i}z)\frac{\partial}{\partial r}K_{0(\zeta_{0i},r)}$$

$$-\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\Big[\hat{C}_n I_{1(\frac{n\pi}{L}r)} + \hat{D}_n K_{1(\frac{n\pi}{L}r)}$$

$$+\frac{L}{2n\pi}\hat{A}_n r I_{0(\frac{n\pi}{L}r)} + \frac{L}{2n\pi}\hat{B}_n r K_{0(\frac{n\pi}{L}r)}\Big]\sin\frac{n\pi z}{L}$$

First partial derivatives of $\xi_{(r,z)}$ with respect to z and second partial derivative with respect to r:

$$\frac{\partial}{\partial z}\frac{\partial^2}{\partial r^2}\xi_{(r,z)} = -\pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^4 J_0^2{}_{(\zeta_{0i}a)}}{J_0^2{}_{(\zeta_{0i}a)} - J_0^2{}_{(\zeta_{0i}b)}}\big[\zeta_{0i}A_i\cosh(\zeta_{0i}z) \tag{4.163}$$

$$+C_i\big(\zeta_{0i}z\sinh(\zeta_{0i}z) + \cosh(\zeta_{0i}z)\big)\big]K_{0(\zeta_{0i},r)}$$

$$-\pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^2 J_0^2{}_{(\zeta_{0i}a)}}{J_0^2{}_{(\zeta_{0i}a)} - J_0^2{}_{(\zeta_{0i}b)}}\big[\zeta_{0i}A_i\cosh(\zeta_{0i}z)$$

$$+C_i\big(\zeta_{0i}z\sinh(\zeta_{0i}z) + \cosh(\zeta_{0i}z)\big)\big]\frac{1}{r}\frac{\partial}{\partial r}K_{0(\zeta_{0i},r)}$$

$$+\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\Big[\hat{C}_n I_{0(\frac{n\pi}{L}r)} + \hat{D}_n K_{0(\frac{n\pi}{L}r)}$$

$$+\frac{L}{2n\pi}\hat{A}_n r I_{1(\frac{n\pi}{L}r)} + \frac{L}{2n\pi}\hat{B}_n r K_{1(\frac{n\pi}{L}r)}\Big]\cos\frac{n\pi z}{L}$$

$$-\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^3}\frac{1}{r}\Big[\hat{C}_n I_{1(\frac{n\pi}{L}r)} + \hat{D}_n K_{1(\frac{n\pi}{L}r)}\Big]\cos\frac{n\pi z}{L}$$

$$+\sum_{n=1}^{\infty}\frac{n\pi}{2L^2}\Big[\hat{A}_n I_{0(\frac{n\pi}{L}r)} + \hat{B}_n K_{0(\frac{n\pi}{L}r)}\Big]\cos\frac{n\pi z}{L}$$

Note the difference between the Fourier Bessel kernel $K_{0(\zeta_{0i},r)}$ with two arguments, and the modified Bessel function of the second kind $K_{0(\frac{n\pi}{L}r)}$ with one argument. Now, using $K_{0(\zeta_{0i},b)} = K_{0(\zeta_{0i},a)} = 0$, the following evaluations are performed.

Eq. (4.159) at $r=a$:

$$\frac{\partial}{\partial z}\nabla^2\xi_{(r,z)}\bigg|_{r=a}=\sum_{n=1}^{\infty}\frac{n\pi}{L^2}\left[\hat{A}_n I_{0(\frac{n\pi}{L}a)}+\hat{B}_n K_{0(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L} \tag{4.164}$$

Eq. (4.161) at $r=a$:

$$\frac{\partial^3}{\partial z^3}\xi_{(r,z)}\bigg|_{r=a}=-\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\bigg[\hat{C}_n I_{0(\frac{n\pi}{L}a)}+\hat{D}_n K_{0(\frac{n\pi}{L}a)} \tag{4.165}$$

$$+\frac{L}{2n\pi}\hat{A}_n a I_{1(\frac{n\pi}{L}a)}+\frac{L}{2n\pi}\hat{B}_n a K_{1(\frac{n\pi}{L}a)}\bigg]\cos\frac{n\pi z}{L}$$

Eq. (4.159) at $r=b$:

$$\frac{\partial}{\partial z}\nabla^2\xi_{(r,z)}\bigg|_{r=b}=\sum_{n=1}^{\infty}\frac{n\pi}{L^2}\left[\hat{A}_n I_{0(\frac{n\pi}{L}b)}+\hat{B}_n K_{0(\frac{n\pi}{L}b)}\right]\cos\frac{n\pi z}{L} \tag{4.166}$$

Eq. (4.161) at $r=b$:

$$\frac{\partial^3}{\partial z^3}\xi_{(r,z)}\bigg|_{r=b}=-\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\bigg[\hat{C}_n I_{0(\frac{n\pi}{L}b)}+\hat{D}_n K_{0(\frac{n\pi}{L}b)} \tag{4.167}$$

$$+\frac{L}{2n\pi}\hat{A}_n b I_{1(\frac{n\pi}{L}b)}+\frac{L}{2n\pi}\hat{B}_n b K_{1(\frac{n\pi}{L}b)}\bigg]\cos\frac{n\pi z}{L}$$

From the equations

$$\frac{\partial}{\partial r}K_{0(\zeta_{0i},r)}\bigg|_{r=a}=-\frac{2}{\pi a}\frac{J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}}$$

and

$$\frac{\partial}{\partial r}K_{0(\zeta_{0i},r)}\bigg|_{r=b}=-\frac{2}{\pi b}$$

at $r=a$, Eq. (4.160) becomes

$$\frac{\partial}{\partial r}\nabla^2\xi_{(r,z)}\bigg|_{r=a} = -\frac{\pi}{a}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}C_i\sinh(\zeta_{0i}z) \tag{4.168}$$

$$+\sum_{n=1}^{\infty}\frac{n\pi}{L^2}\left[\hat{A}_n I_{1(\frac{n\pi}{L}a)} + \hat{B}_n K_{1(\frac{n\pi}{L}a)}\right]\sin\frac{n\pi z}{L},$$

and at $r = b$ it is

$$\frac{\partial}{\partial r}\nabla^2\xi_{(r,z)}\bigg|_{r=b} = -\frac{\pi}{b}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}C_i\sinh(\zeta_{0i}z) \tag{4.169}$$

$$+\sum_{n=1}^{\infty}\frac{n\pi}{L^2}\left[\hat{A}_n I_{1(\frac{n\pi}{L}b)} + \hat{B}_n K_{1(\frac{n\pi}{L}b)}\right]\sin\frac{n\pi z}{L}.$$

With the same substitution at $r = a$ Eq. (4.162) generates

$$\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\xi_{(r,z)}\bigg|_{r=a} = -\frac{\pi}{a}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^4 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\left[A_i\sinh(\zeta_{0i}z)+C_i z\cosh(\zeta_{0i}z)\right] \tag{4.170}$$

$$-\frac{\pi}{a}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}C_i\sinh(\zeta_{0i}z)$$

$$-\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\Big[\hat{C}_n I_{1(\frac{n\pi}{L}a)} + \hat{D}_n K_{1(\frac{n\pi}{L}a)}$$

$$+\frac{L}{2n\pi}\hat{A}_n a I_{0(\frac{n\pi}{L}a)} + \frac{L}{2n\pi}\hat{B}_n a K_{0(\frac{n\pi}{L}a)}\Big]\sin\frac{n\pi z}{L},$$

and Eq. (4.163) gives

$$\frac{\partial}{\partial z}\frac{\partial^2}{\partial r^2}\xi_{(r,z)}\bigg|_{r=a} = \frac{\pi}{a^2}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\Big[\zeta_{0i}A_i\cosh(\zeta_{0i}z) \tag{4.171}$$

$$+C_i\big(\zeta_{0i}z\sinh(\zeta_{0i}z)+\cosh(\zeta_{0i}z)\big)\Big]$$

$$+\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\Big[\hat{C}_n I_{0(\frac{n\pi}{L}a)} + \hat{D}_n K_{0(\frac{n\pi}{L}a)}$$

$$+\frac{L}{2n\pi}\hat{A}_n a I_{1(\frac{n\pi}{L}a)} + \frac{L}{2n\pi}\hat{B}_n a K_{1(\frac{n\pi}{L}a)}\Big]\cos\frac{n\pi z}{L}$$

$$-\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^3}\frac{1}{a}\left[\hat{C}_n I_{1(\frac{n\pi}{L}a)} + \hat{D}_n K_{1(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L}$$

$$+\sum_{n=1}^{\infty}\frac{n\pi}{2L^2}\left[\hat{A}_n I_{0(\frac{n\pi}{L}a)} + \hat{B}_n K_{0(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L}.$$

Moreover, exploiting the same substitution, at $r = b$ Eq. (4.162) transfers to

$$\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\xi_{(r,z)}\bigg|_{r=b} = -\frac{\pi}{b}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^4 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}\left[A_i \sinh(\zeta_{0i}z) + C_i z\cosh(\zeta_{0i}z)\right] \tag{4.172}$$
$$-\frac{\pi}{b}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}C_i \sinh(\zeta_{0i}z)$$
$$-\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\left[\hat{C}_n I_1(\tfrac{n\pi}{L}b) + \hat{D}_n K_1(\tfrac{n\pi}{L}b)\right.$$
$$\left. + \frac{L}{2n\pi}\hat{A}_n b I_0(\tfrac{n\pi}{L}b) + \frac{L}{2n\pi}\hat{B}_n b K_0(\tfrac{n\pi}{L}b)\right]\sin\frac{n\pi z}{L},$$

and Eq. (4.163) becomes

$$\frac{\partial}{\partial z}\frac{\partial^2}{\partial r^2}\xi_{(r,z)}\bigg|_{r=b} = \frac{\pi}{b^2}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}\left[\zeta_{0i}A_i \cosh(\zeta_{0i}z)\right. \tag{4.173}$$
$$\left. + C_i\left(\zeta_{0i}z\sinh(\zeta_{0i}z) + \cosh(\zeta_{0i}z)\right)\right]$$
$$+\sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\left[\hat{C}_n I_0(\tfrac{n\pi}{L}b) + \hat{D}_n K_0(\tfrac{n\pi}{L}b)\right.$$
$$\left. + \frac{L}{2n\pi}\hat{A}_n b I_1(\tfrac{n\pi}{L}b) + \frac{L}{2n\pi}\hat{B}_n b K_1(\tfrac{n\pi}{L}b)\right]\cos\frac{n\pi z}{L}$$
$$-\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^3}\frac{1}{b}\left[\hat{C}_n I_1(\tfrac{n\pi}{L}b) + \hat{D}_n K_1(\tfrac{n\pi}{L}b)\right]\cos\frac{n\pi z}{L}$$
$$+\sum_{n=1}^{\infty}\frac{n\pi}{2L^2}\left[\hat{A}_n I_0(\tfrac{n\pi}{L}b) + \hat{B}_n K_0(\tfrac{n\pi}{L}b)\right]\cos\frac{n\pi z}{L}.$$

Now, letting $z = L$ results in the following equations.

Eq. (4.159):

$$\frac{\partial}{\partial z}\nabla^2\xi_{(r,z)}\bigg|_{z=L} = \pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^4 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}C_i \cosh(\zeta_{0i}L)K_0(\zeta_{0i}r) \tag{4.174}$$
$$+\sum_{n=1}^{\infty}\frac{n\pi}{L^2}\left[\hat{A}_n I_0(\tfrac{n\pi}{L}r) + \hat{B}_n K_0(\tfrac{n\pi}{L}r)\right](-1)^n$$

Eq. (4.160):

$$\frac{\partial}{\partial r}\nabla^2\xi_{(r,z)}\bigg|_{z=L} = \pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}C_i \sinh(\zeta_{0i}L)\frac{\partial}{\partial r}K_0(\zeta_{0i}r) \tag{4.175}$$

Eq. (4.161):

$$\left.\frac{\partial^3}{\partial z^3}\xi_{(r,z)}\right|_{z=L} = \pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^5 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\left[A_i\cosh(\zeta_{0i}L)+C_iL\sinh(\zeta_{0i}L)\right]K_0(\zeta_{0i},r) + \pi^2\sum_{i=1}^{\infty}\frac{3\zeta_{0i}^4 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}C_i\cosh(\zeta_{0i}L)K_0(\zeta_{0i},r) - \sum_{n=1}^{\infty}\frac{n^3\pi^3}{L^4}\left[\hat{C}_nI_0(\tfrac{n\pi}{L}r)+\hat{D}_nK_0(\tfrac{n\pi}{L}r)+\frac{L}{2n\pi}\hat{A}_nrI_1(\tfrac{n\pi}{L}r)+\frac{L}{2n\pi}\hat{B}_nrK_1(\tfrac{n\pi}{L}r)\right](-1)^n \tag{4.176}$$

Eq. (4.162):

$$\left.\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\xi_{(r,z)}\right|_{z=L} = \pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^4 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\left[A_i\sinh(\zeta_{0i}L)+C_iL\cosh(\zeta_{0i}L)\right]\frac{\partial}{\partial r}K_0(\zeta_{0i},r) + \pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}C_i\sinh(\zeta_{0i}L)\frac{\partial}{\partial r}K_0(\zeta_{0i},r) \tag{4.177}$$

Applying the shear stress boundary condition at $z = L$ and by using Eqs. (4.175) and (4.177), we may write

$$(1-\nu)\pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}C_i\sinh(\zeta_{0i}L)\frac{\partial}{\partial r}K_0(\zeta_{0i},r) - \pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^4 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\left[A_i\sinh(\zeta_{0i}L)+C_iL\cosh(\zeta_{0i}L)\right]\frac{\partial}{\partial r}K_0(\zeta_{0i},r) - \pi^2\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}C_i\sinh(\zeta_{0i}L)\frac{\partial}{\partial r}K_0(\zeta_{0i},r) = 0 \tag{4.178}$$

or

$$-\pi^2\sum_{i=1}^{\infty}\frac{\zeta_{0i}^3 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a)-J_0^2(\zeta_{0i}b)}\left\{2\nu C_i\sinh(\zeta_{0i}L)+\zeta_{0i}\left[A_i\sinh(\zeta_{0i}L)+C_iL\cosh(\zeta_{0i}L)\right]\right\}\frac{\partial}{\partial r}K_0(\zeta_{0i},r) = 0. \tag{4.179}$$

The summation in Eq. (4.179) is zero if

$$2\nu C_i \sinh(\zeta_{0i}L) + \zeta_{0i}\left[A_i \sinh(\zeta_{0i}L) + C_i L\cosh(\zeta_{0i}L)\right] = 0 \tag{4.180}$$

or

$$\left[2\nu + \zeta_{0i}L\coth(\zeta_{0i}L)\right]C_i + \zeta_{0i}A_i = 0. \tag{4.181}$$

Hence, C_i is expressed by

$$C_i = \omega_{(\zeta_{oi})}A_i \tag{4.182}$$

where $\coth(\zeta_{0i}L)$ is hyperbolic cotangent and $\omega_{(\zeta_{oi})}$ is defined by Eq. (4.183).

$$\omega_{(\zeta_{oi})} = \frac{-\zeta_{0i}}{2\nu + \zeta_{0i}L\coth(\zeta_{0i}L)} \tag{4.183}$$

Substituting Eqs. (4.164) and (4.171) into the radial stress boundary condition at $r = a$ and using the orthogonality property of cosine yields

$$\begin{aligned}
&\Bigg\{\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}I_{0(\frac{n\pi}{L}a)} - \frac{n^2\pi^2}{2L^3}aI_{1(\frac{n\pi}{L}a)}\right]\hat{A}_n \\
&+\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}K_{0(\frac{n\pi}{L}a)} - \frac{n^2\pi^2}{2L^3}aK_{1(\frac{n\pi}{L}a)}\right]\hat{B}_n \\
&+\left[-\frac{n^3\pi^3}{L^4}I_{0(\frac{n\pi}{L}a)} + \frac{n^2\pi^2}{L^3}\frac{1}{a}I_{1(\frac{n\pi}{L}a)}\right]\hat{C}_n \\
&+\left[-\frac{n^3\pi^3}{L^4}K_{0(\frac{n\pi}{L}a)} + \frac{n^2\pi^2}{L^3}\frac{1}{a}K_{1(\frac{n\pi}{L}a)}\right]\hat{D}_n \\
&-\frac{\pi}{a^2}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\Bigg[\zeta_{0i}A_i\int_{-L}^{L}\cos\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz \\
&\qquad + C_i\Bigg(\zeta_{0i}\int_{-L}^{L}z\cos\frac{n\pi z}{L}\sinh(\zeta_{0i}z)dz \\
&\qquad\qquad + \int_{-L}^{L}\cos\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz\Bigg)\Bigg]\Bigg\} = -\wp_{1(n)}(1-\nu).
\end{aligned} \tag{4.184}$$

The integrals in Eq. (4.183) are evaluated from Eqs. (4.185) and (4.186).

$$\int_{-L}^{L} z\cos\frac{n\pi z}{L}\sinh(\zeta_{0i}z)dz = \frac{2L^2(-1)^n}{\left(\zeta_{0i}^2L^2+n^2\pi^2\right)^2}\left[-\left(\zeta_{0i}^2L^2-n^2\pi^2\right)\sinh(\zeta_{0i}L)\right. \quad (4.185)$$

$$\left.+\left(\zeta_{0i}^2L^2+n^2\pi^2\right)\zeta_{0i}L\cosh(\zeta_{0i}L)\right] = \Omega_{1(\zeta_{0i},n)}$$

$$\int_{-L}^{L}\cos\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz = \frac{2\zeta_{0i}L^2(-1)^n}{\left(\zeta_{0i}^2L^2+n^2\pi^2\right)}\sinh(\zeta_{0i}L) = \Omega_{2(\zeta_{0i},n)} \quad (4.186)$$

Incorporating Eqs. (4.182), (4.185), (4.186) into Eq. (4.184), gives

$$\lambda_{11(n)}\hat{A}_n + \lambda_{12(n)}\hat{B}_n + \lambda_{13(n)}\hat{C}_n + \lambda_{14(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{1(\zeta_{0i},n)}A_i = \wp_{1(n)} \quad (4.187)$$

where $\lambda_{11(n)}$, $\lambda_{12(n)}$, $\lambda_{13(n)}$ and $\lambda_{14(n)}$ are given by Eq. (4.188)

$$\lambda_{11(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}I_{0(\frac{n\pi}{L}a)} - \frac{n^2\pi^2}{2L^3}aI_{1(\frac{n\pi}{L}a)}\right] \quad (4.188)$$

$$\lambda_{12(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}K_{0(\frac{n\pi}{L}a)} - \frac{n^2\pi^2}{2L^3}aK_{1(\frac{n\pi}{L}a)}\right]$$

$$\lambda_{13(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}I_{0(\frac{n\pi}{L}a)} + \frac{n^2\pi^2}{L^3}\frac{1}{a}I_{1(\frac{n\pi}{L}a)}\right]$$

$$\lambda_{14(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}K_{0(\frac{n\pi}{L}a)} + \frac{n^2\pi^2}{L^3}\frac{1}{a}K_{1(\frac{n\pi}{L}a)}\right],$$

and $\Lambda_{1(\zeta_{oi},n)}$ is defined by Eq. (4.189).

$$\Lambda_{1(\zeta_{oi},n)} = \frac{1}{(1-\nu)}\frac{\pi}{a^2}\frac{2\zeta_{0i}^2J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\left[\zeta_{0i}\Omega_{2(\zeta_{oi},n)}\right. \quad (4.189)$$

$$\left.+\,\omega_{(\zeta_{oi})}\left(\zeta_{0i}\Omega_{1(\zeta_{oi},n)} + \Omega_{2(\zeta_{oi},n)}\right)\right]$$

Enforcing other radial boundary conditions, Eqs. (4.126), (4.127) and (4.128), leads to *

$$\lambda_{21(n)}\hat{A}_n + \lambda_{22(n)}\hat{B}_n + \lambda_{23(n)}\hat{C}_n + \lambda_{24(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{2(\zeta_{0i},n)}A_i = \wp_{2(n)} \quad (4.190)$$

* See Appendix F for details.

$$\lambda_{31(n)}\hat{A}_n + \lambda_{32(n)}\hat{B}_n + \lambda_{33(n)}\hat{C}_n + \lambda_{34(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{3(\zeta_{0i},n)}A_i = \wp_{3(n)} \tag{4.191}$$

$$\lambda_{41(n)}\hat{A}_n + \lambda_{42(n)}\hat{B}_n + \lambda_{43(n)}\hat{C}_n + \lambda_{44(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{4(\zeta_{0i},n)}A_i = \wp_{4(n)} \tag{4.192}$$

where $\lambda_{21(n)}$, $\lambda_{22(n)}$, $\lambda_{23(n)}$, $\lambda_{24(n)}$ and $\Lambda_{2(\zeta_{oi},n)}$ are given by Eq. (4.193) and (4.194),

$$\lambda_{21(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}I_{0(\frac{n\pi}{L}b)} - \frac{n^2\pi^2}{2L^3}bI_{1(\frac{n\pi}{L}b)}\right] \tag{4.193}$$

$$\lambda_{22(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu-\frac{1}{2}\right)\frac{n\pi}{L^2}K_{0(\frac{n\pi}{L}b)} - \frac{n^2\pi^2}{2L^3}bK_{1(\frac{n\pi}{L}b)}\right]$$

$$\lambda_{23(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}I_{0(\frac{n\pi}{L}b)} + \frac{n^2\pi^2}{L^3}\frac{1}{b}I_{1(\frac{n\pi}{L}b)}\right]$$

$$\lambda_{24(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}K_{0(\frac{n\pi}{L}b)} + \frac{n^2\pi^2}{L^3}\frac{1}{b}K_{1(\frac{n\pi}{L}b)}\right]$$

$$\Lambda_{2(\zeta_{oi},n)} = \frac{1}{(1-\nu)}\frac{\pi}{b^2}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\Big[\zeta_{0i}\Omega_{2(\zeta_{oi},n)} + \omega_{(\zeta_{oi})}\left(\zeta_{0i}\Omega_{1(\zeta_{oi},n)} + \Omega_{2(\zeta_{oi},n)}\right)\Big], \tag{4.194}$$

and $\lambda_{31(n)}$, $\lambda_{32(n)}$, $\lambda_{33(n)}$, $\lambda_{34(n)}$ and $\Lambda_{3(\zeta_{oi},n)}$ are expressed by Eq. (4.195) and (4.196),

$$\lambda_{31(n)} = -\left[\frac{n\pi}{L^2}I_{1(\frac{n\pi}{L}a)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}aI_{0(\frac{n\pi}{L}a)}\right] \tag{4.195}$$

$$\lambda_{32(n)} = -\left[\frac{n\pi}{L^2}K_{1(\frac{n\pi}{L}a)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}aK_{0(\frac{n\pi}{L}a)}\right]$$

$$\lambda_{33(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}I_{1(\frac{n\pi}{L}a)}\right]$$

$$\lambda_{34(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}K_{1(\frac{n\pi}{L}a)}\right]$$

$$\Lambda_{3(\zeta_{oi},n)} = \frac{1}{1-\nu}\frac{\pi}{a}\frac{2\zeta_{0i}^{3}J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^{2}-J_{0(\zeta_{0i}b)}^{2}}\Big[2\nu\omega_{(\zeta_{oi})}\Omega_{4(\zeta_{oi},n)} - \zeta_{0i}\big(\Omega_{4(\zeta_{oi},n)}+\omega_{(\zeta_{oi})}\Omega_{3(\zeta_{oi},n)}\big)\Big], \tag{4.196}$$

and finally $\Lambda_{4(\zeta_{oi},n)}$, $\lambda_{41(n)}$, $\lambda_{42(n)}$, $\lambda_{43(n)}$, and $\lambda_{44(n)}$ are defined by Eqs. (4.197) and (4.198).

$$\Lambda_{4(\zeta_{oi},n)} = \frac{1}{1-\nu}\frac{\pi}{b}\frac{2\zeta_{0i}^{3}J_{0(\zeta_{0i}a))}^{2}}{J_{0(\zeta_{0i}a)}^{2}-J_{0(\zeta_{0i}b)}^{2}}\Big[2\nu\omega_{(\zeta_{oi})}\Omega_{4(\zeta_{oi},n)} - \zeta_{0i}\big(\Omega_{4(\zeta_{oi},n)}+\omega_{(\zeta_{oi})}\Omega_{3(\zeta_{oi},n)}\big)\Big] \tag{4.197}$$

$$\lambda_{41(n)} = -\left[\frac{n\pi}{L^2}I_{1(\frac{n\pi}{L}b)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}bI_{0(\frac{n\pi}{L}b)}\right] \tag{4.198}$$

$$\lambda_{42(n)} = -\left[\frac{n\pi}{L^2}K_{1(\frac{n\pi}{L}b)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}bK_{0(\frac{n\pi}{L}b)}\right]$$

$$\lambda_{43(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}I_{1(\frac{n\pi}{L}b)}\right]$$

$$\lambda_{44(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}K_{1(\frac{n\pi}{L}b)}\right]$$

The definitions for $\Omega_{3(\zeta_{0i},n)}$ and $\Omega_{4(\zeta_{0i},n)}$ are given by Eqs. (4.199) and (4.200), respectively.

$$\int_{-L}^{L} z\sin\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz = \frac{2L^2(-1)^n}{\left(\zeta_{0i}^2L^2+n^2\pi^2\right)^2}\Big[2n\pi L\zeta_{0i}\sinh(\zeta_{0i}L) - \left(\zeta_{0i}^2L^2+n^2\pi^2\right)n\pi\cosh(\zeta_{0i}L)\Big] = \Omega_{3(\zeta_{0i},n)} \tag{4.199}$$

$$\int_{-L}^{L}\sin\frac{n\pi z}{L}\sinh(\zeta_{0i}z)dz = \frac{-2n\pi L(-1)^n}{\left(\zeta_{0i}^2L^2+n^2\pi^2\right)}\sinh(\zeta_{0i}L) = \Omega_{4(\zeta_{0i},n)} \tag{4.200}$$

By applying the axial stress boundary condition and using Eqs. (4.174) and (4.176), we may write

$$2\zeta_{0i}^2\left[(1-2\nu)\omega(\zeta_{oi})\cosh(\zeta_{0i}L)-\zeta_{0i}\cosh(\zeta_{0i}L)-\zeta_{0i}\omega(\zeta_{oi})L\sinh(\zeta_{0i}L)\right]A_i \quad (4.201)$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{n\pi}{L^2}\left[(2-\nu)\int_a^b rK_0(\zeta_{0i},r)I_0(\tfrac{n\pi}{L}r)dr+\frac{n\pi}{2L^2}\int_a^b r^2K_0(\zeta_{0i},r)I_1(\tfrac{n\pi}{L}r)dr\right]\hat{A}_n$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{n\pi}{L^2}\left[(2-\nu)\int_a^b rK_0(\zeta_{0i},r)K_0(\tfrac{n\pi}{L}r)dr+\frac{n\pi}{2L^2}\int_a^b r^2K_0(\zeta_{0i},r)K_1(\tfrac{n\pi}{L}r)dr\right]\hat{B}_n$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{n^3\pi^3}{L^4}\left[\int_a^b rK_0(\zeta_{0i},r)I_0(\tfrac{n\pi}{L}r)dr\right]\hat{C}_n$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{n^3\pi^3}{L^4}\left[\int_a^b rK_0(\zeta_{0i},r)K_0(\tfrac{n\pi}{L}r)dr\right]\hat{D}_n$$

$$=-(1-\nu)\left\{\frac{2}{\pi^2\zeta_{0i}^2}\left[1-\left(\frac{J_n(\zeta_{0i}b)}{J_n(\zeta_{0i}a)}\right)^2\right]\wp_5(\zeta_{0i})+\wp_6(\zeta_{1i})\int_a^b rK_0(\zeta_{0i},r)K_{01}(\zeta_{1i},r)dr\right\}.$$

By introducing

$$\Lambda_5(\zeta_{0i})=2\zeta_{0i}^2\left[(1-2\nu)\omega(\zeta_{oi})\cosh(\zeta_{0i}L)-\zeta_{0i}\cosh(\zeta_{0i}L)-\zeta_{0i}\omega(\zeta_{oi})L\sinh(\zeta_{0i}L)\right] \quad (4.202)$$

and

$$\lambda_{51}(\zeta_{0i},n)=(-1)^n\frac{n\pi}{L^2}\left[(2-\nu)\int_a^b rK_0(\zeta_{0i},r)I_0(\tfrac{n\pi}{L}r)dr+\frac{n\pi}{2L^2}\int_a^b r^2K_0(\zeta_{0i},r)I_1(\tfrac{n\pi}{L}r)dr\right] \quad (4.203)$$

$$\lambda_{52}(\zeta_{0i},n)=(-1)^n\frac{n\pi}{L^2}\left[(2-\nu)\int_a^b rK_0(\zeta_{0i},r)K_0(\tfrac{n\pi}{L}r)dr+\frac{n\pi}{2L^2}\int_a^b r^2K_0(\zeta_{0i},r)K_1(\tfrac{n\pi}{L}r)dr\right]$$

$$\lambda_{53}(\zeta_{0i},n)=(-1)^n\frac{n^3\pi^3}{L^4}\left[\int_a^b rK_0(\zeta_{0i},r)I_0(\tfrac{n\pi}{L}r)dr\right]$$

$$\lambda_{54}(\zeta_{0i},n)=(-1)^n\frac{n^3\pi^3}{L^4}\left[\int_a^b rK_0(\zeta_{0i},r)K_0(\tfrac{n\pi}{L}r)dr\right]$$

$$\Gamma_{5}(\zeta_{0i},\zeta_{1i}) = -(1-\nu)\left\{\frac{2}{\pi^2\zeta_{0i}^2}\left[1-\left(\frac{J_0(\zeta_{0i}b)}{J_0(\zeta_{0i}a)}\right)^2\right]\wp_5(\zeta_{0i})\right. \tag{4.204}$$

$$\left. + \wp_6(\zeta_{1i})\int_a^b rK_0(\zeta_{0i},r)K_{01}(\zeta_{1i},r)dr\right\}$$

Eq. (4.200) is reduced to

$$\Lambda_5(\zeta_{0i})A_i + \sum_{n=1}^{\infty}\Big[\lambda_{51}(\zeta_{0i},n)\hat{A}_n + \lambda_{52}(\zeta_{0i},n)\hat{B}_n \tag{4.205}$$

$$+ \lambda_{53}(\zeta_{0i},n)\hat{C}_n + \lambda_{54}(\zeta_{0i},n)\hat{D}_n\Big] = \Gamma_5(\zeta_{0i},\zeta_{1i}).$$

Eqs. (4.187), (4.190), (4.191), (4.192) and (4.205) represent a system of equations where unknowns are $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$, $\hat{D}_n$ and A_i. As it is practiced in numerical simulations, the infinite series is changed to a finite summations with an acceptable truncation error. Hence, the upper limit for i and n are changed to M and N, respectively. Expanding these finite summations would result in a system of equations with $4N+M$ unknowns and equations. Unknowns are $\hat{A}_1\cdots\hat{A}_n$, $\hat{B}_1\cdots\hat{B}_n$, $\hat{C}_1\cdots\hat{C}_n$, $\hat{D}_1\cdots\hat{D}_n$ and $A_1\cdots A_n$. Eq. (4.204) gives M equations by letting i vary from 1 to M. Moreover, allowing n to advance from 1 to N in Eqs. (4.187), (4.190), (4.191) and (4.192), results in $4N$ equations. The coefficient matrix for this system of equations is shown in Fig. 4.2. By solving this system of equations, the arbitrary constants for the complementary solution of the axial stress function are obtained. The combination of the complementary and the Green's function solutions for the axial stress function yields a solution that satisfies both the boundary conditions and the axial body force. This solution together with the radial Green's function determine the disturbance of stress and strain in the given coil.

$$
\left[\begin{array}{cccccccccccccccccc}
\lambda_{11}(1) & \lambda_{12}(1) & \lambda_{13}(1) & \lambda_{14}(1) & & & & & & & & & & \Lambda_1(\zeta_{01},1) & \cdots & \Lambda_1(\zeta_{0q},1) & \cdots & \Lambda_1(\zeta_{0M},1) \\
\lambda_{21}(1) & \lambda_{22}(1) & \lambda_{23}(1) & \lambda_{24}(1) & & & & & & & & & & \Lambda_2(\zeta_{01},1) & \cdots & \Lambda_2(\zeta_{0q},1) & \cdots & \Lambda_2(\zeta_{0M},1) \\
\lambda_{31}(1) & \lambda_{32}(1) & \lambda_{33}(1) & \lambda_{34}(1) & & & & & & & & & & \Lambda_3(\zeta_{01},1) & \cdots & \Lambda_3(\zeta_{0q},1) & \cdots & \Lambda_3(\zeta_{0M},1) \\
\lambda_{41}(1) & \lambda_{42}(1) & \lambda_{43}(1) & \lambda_{44}(1) & & & & & & & & & & \Lambda_4(\zeta_{01},1) & \cdots & \Lambda_4(\zeta_{0q},1) & \cdots & \Lambda_4(\zeta_{0M},1) \\
 & & & & & \lambda_{11}(p) & \lambda_{12}(p) & \lambda_{13}(p) & \lambda_{14}(p) & & & & & \Lambda_1(\zeta_{01},p) & \cdots & \Lambda_1(\zeta_{0q},p) & \cdots & \Lambda_1(\zeta_{0M},p) \\
 & & & & & \lambda_{21}(p) & \lambda_{22}(p) & \lambda_{23}(p) & \lambda_{24}(p) & & & & & \Lambda_2(\zeta_{01},p) & \cdots & \Lambda_2(\zeta_{0q},p) & \cdots & \Lambda_2(\zeta_{0M},p) \\
 & & & & & \lambda_{31}(p) & \lambda_{32}(p) & \lambda_{33}(p) & \lambda_{34}(p) & & & & & \Lambda_3(\zeta_{01},p) & \cdots & \Lambda_3(\zeta_{0q},p) & \cdots & \Lambda_3(\zeta_{0M},p) \\
 & & & & & \lambda_{41}(p) & \lambda_{42}(p) & \lambda_{43}(p) & \lambda_{44}(p) & & & & & \Lambda_4(\zeta_{01},p) & \cdots & \Lambda_4(\zeta_{0q},p) & \cdots & \Lambda_4(\zeta_{0M},p) \\
 & & & & & & & & & & \lambda_{11}(N) & \cdots & \lambda_{14}(N) & \Lambda_1(\zeta_{01},N) & \cdots & \Lambda_1(\zeta_{0q},N) & \cdots & \Lambda_1(\zeta_{0M},N) \\
 & & & & & & & & & & \lambda_{21}(N) & \cdots & \lambda_{24}(N) & \Lambda_2(\zeta_{01},N) & \cdots & \Lambda_2(\zeta_{0q},N) & \cdots & \Lambda_2(\zeta_{0M},N) \\
 & & & & & & & & & & \lambda_{31}(N) & \cdots & \lambda_{34}(N) & \Lambda_3(\zeta_{01},N) & \cdots & \Lambda_3(\zeta_{0q},N) & \cdots & \Lambda_3(\zeta_{0M},N) \\
 & & & & & & & & & & \lambda_{41}(N) & \cdots & \lambda_{44}(N) & \Lambda_4(\zeta_{01},N) & \cdots & \Lambda_4(\zeta_{0q},N) & \cdots & \Lambda_4(\zeta_{0M},N) \\
\lambda_{51}(\zeta_{01},1) & \lambda_{52}(\zeta_{01},1) & \lambda_{53}(\zeta_{01},1) & \lambda_{54}(\zeta_{01},1) & \cdots & \lambda_{51}(\zeta_{01},p) & \lambda_{52}(\zeta_{01},p) & \lambda_{53}(\zeta_{01},p) & \lambda_{54}(\zeta_{01},p) & \cdots & \lambda_{51}(\zeta_{01},N) & \cdots & \lambda_{54}(\zeta_{01},N) & \Lambda_5(\zeta_{01}) & & & & \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & & & & \\
\lambda_{51}(\zeta_{0q},1) & \lambda_{52}(\zeta_{0q},1) & \lambda_{53}(\zeta_{0q},1) & \lambda_{54}(\zeta_{0q},1) & \cdots & \lambda_{51}(\zeta_{0q},p) & \lambda_{52}(\zeta_{0q},p) & \lambda_{53}(\zeta_{0q},p) & \lambda_{54}(\zeta_{0q},p) & \cdots & \lambda_{51}(\zeta_{0q},N) & \cdots & \lambda_{54}(\zeta_{0q},N) & & & \Lambda_5(\zeta_{0q}) & & \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & & & & \\
\lambda_{51}(\zeta_{0M},1) & \lambda_{52}(\zeta_{0M},1) & \lambda_{53}(\zeta_{0M},1) & \lambda_{54}(\zeta_{0M},1) & \cdots & \lambda_{51}(\zeta_{0M},p) & \lambda_{52}(\zeta_{0M},p) & \lambda_{53}(\zeta_{0M},p) & \lambda_{54}(\zeta_{0M},p) & \cdots & \lambda_{51}(\zeta_{0M},N) & \cdots & \lambda_{54}(\zeta_{0M},N) & & & & & \Lambda_5(\zeta_{0M})
\end{array}\right]
$$

Figure 4.2 Coefficient matrix of the system of equations for arbitrary constants.

CHAPTER 5
RESULTS AND DISCUSSIONS

Introduction

Results of the generalized plane strain analysis and Green's functions solution are presented. A comparison of the analytical results with the finite element results is performed to check the validity of the solutions. The finite element results are obtained* through the ABAQUS[30] finite element solver. Linked with the finite element solver is an in-house FORTRAN subroutine[31] which determines the non-uniform body forces created by the magnetic fields. These forces are calculated at each gaussian integration point and are used as an input to the finite element program.

Results of Generalized Plane Strain Analysis

An example magnet is generated to examine the results of a generalized plane strain analysis. The example design is for a 15 Tesla magnet with two coils each with two layers (Fig. 5.1). The parameters for each of the coils are given in Table 5.1, where for each layer the inside radius is a_1, the outside radius is a_2, the half length is L, and the current density is J. The outer radius of the reinforcement, where applicable, is given by a_3.

Table 5.1 Parameters for 15 T example magnet with two coils, each with two layers.

Coil	a_1 (mm)	a_2 (mm)	a_3 (mm)	L (mm)	J (A/mm^2)
1a	100	126	---	250	102.36
1b	126	148	154	250	161.37
2a	157	170	---	250	155.55
2b	170	210	---	250	149.54

* Finite element analysis was done by Iain Dixon at National High Magnetic Field Laboratory.

The example magnet has two combined layers containing Nb_3Sn conductor in the inner coil, and two connected layers containing NbTi conductor in the outer coil. The second layer of the inner coil contains a reinforcement on the outer diameter, as indicated in the Fig. 5.1.

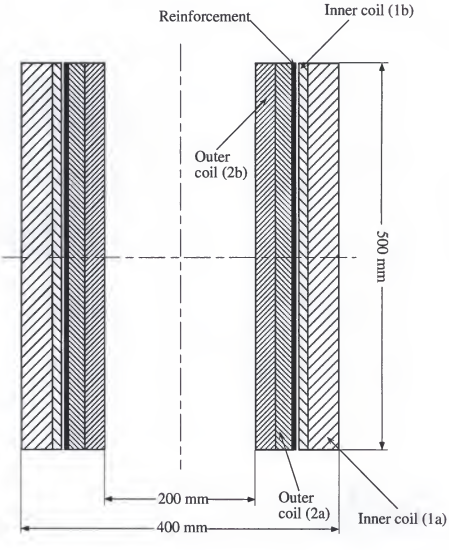

Figure 5.1 15 Tesla example magnet with two coils, each with two layers.

The difference in conductors is reflected in the coil material properties listed in Table 5.2. It is noted that the properties in the tangential direction are dominated by the properties of the conductor. The properties transverse to the conductor, in the r and z

directions, are influenced strongly by the properties of the insulation between the conductors.

Table 5.2 Materials properties of coils and reinforcement.

	Nb_3Sn Coils	NbTi Coils	Reinforcement
E_θ (GPa)	66.500	81.500	190.000
E_r (GPa)	41.400	45.400	190.000
E_z (GPa)	44.500	49.400	190.000
$G_{\theta r}$ (GPa)	14.700	16.000	73.000
$G_{\theta z}$ (GPa)	15.700	17.500	73.000
G_{rz} (GPa)	11.800	12.500	73.000
$v_{\theta r}$	0.341	0.341	0.305
$v_{\theta z}$	0.318	0.319	0.305
v_{rz}	0.204	0.182	0.305
$\alpha_\theta \Delta T$	-0.0028	-0.0028	-0.0029
$\alpha_r \Delta T$	-0.0048	-0.0048	-0.0029
$\alpha_z \Delta T$	-0.0048	-0.0048	-0.0029

Magnetic Stress and Strain

The following is a comparison between results generated by the generalized plane strain and the finite element methods. Stresses and strains are calculated for the example magnet. Moreover, plane stress and plane strain analyses are performed for the example magnet and compared along with the generalized plane strain analysis.

In reference to Fig. 5.2, the tangential stress is graphed from the results of the finite element and the generalized plane strain analyses. Small differences are displayed with the greatest discrepancy within the reinforcement. This may be caused by a stiffer solution resulting from relatively fewer elements being used in the narrow reinforcement region.

Fig. 5.3 shows the characteristics of radial stress through the magnet. The difference between generalized plane strain and finite element for radial stress is even smaller. Fig. 5.4 shows the characteristics of axial stress. Since in the finite element

methods the axial strain is not being constrained to a constant, the results of the finite element solution is improved over the generalized plane strain analysis.

Figs. 5.5 and 5.6 are plots of the tangential and radial strains. Both graphs show an extremely close relationship between the two methods. Fig. 5.7 shows the variation of axial strain through each coil for finite element along with the constant value of the axial strain for generalized plane strain analysis. The maximum discrepancy for axial strain between these two methods is nine percent.

The theory of generalized plane strain is compared to other analytical methods as well as finite elements. Using the example magnet, the results of the generalized plane strain analysis are studied along with those determined by plane stress and plane strain assumptions.

The stress in the tangential and radial directions for the three analytical solutions are plotted in Figs. 5.8 and 5.9. Results match quite well and are indistinguishable through the superconducting layers. The reinforcement possesses the largest difference between the generalized plane strain and the other theories.

The greatest improvement in the approximation of stress is shown in Fig. 5.10, where the axial stress for plane strain and generalized plane strain is plotted. Note that for the plane stress condition the axial stress component is zero. With a more general approximation of axial strain, the generalized plane strain theory is able to predict axial stress to a higher degree of accuracy. Referring to Fig. 5.4, it is easily seen that the generalized plane strain matches the finite element more closely than the plane strain method. In fact the plane strain result predicts the axial stress in the tensile regime, whereas the generalized plane strain and finite element analyses show the axial stress in compression. From the direction of the radial and axial fields produced from the magnet, it may be easily found that the axial stress is compressive in nature.

The tangential and radial strains in Figs. 5.11 and 5.12 respectively show the same type of relationships. The strains start to deviate significantly with the plane stress results

being bound by the other theories. The axial strain for plane stress and generalized plane strain is plotted in Fig. 5.13. The plane stress predicts much lower values for axial strain. Note that for the plane strain condition the axial strain component is zero.

Thermal Stress and Strain

In the previous comparisons magnetic forces are considered as the applied loads. Since the generalized plane strain assumption may be applied to the thermal as well as magnetic loads, this case is also studied to observe the effects of cryogenic temperatures on superconducting magnets. The following studies the same 15 Tesla magnet by comparing the stress and strain results determined by the generalized plane strain analysis with the finite element methods. The thermal loads, that are applied in terms of $\alpha\Delta T$, are defined in Table 5.2.

The comparisons that follow are similar in structure to the magnetic load comparisons. The differences between the finite element and generalized plane strain methods are even less because of the near constant behavior of the axial strain. In Figs. 5.14 - 5.18, the tangential, radial, and axial stresses and tangential and radial strains show excellent agreement. The slight differences that do exist, occur more frequently in the first coil than the second. The reason for this may be seen in Fig. 5.19 where there is a larger variation of axial strain through the first coil than the second, which is near zero.

Results of Green's Functions Solution

The Green's functions solution is applied to a 23 T isotropic coil and the results are compared to the results of finite element methods. The parameters for this coil is given in Table 5.3, where the inside radius is a, the outside radius is b, the length of the coil is $2L$, the current density is J, and the elastic modulus and the Poison's ratio for the coil are E and ν, respectively.

Figs. 5.20 and 5.21 show the tangential and radial stresses through the coil along the radius at three different axial position (midplane, half between the midplane and the top, and the top of the coil). Little differences are displayed between results of the Green's functions solution and finite element methods with the greatest discrepancy at the top of the coil. Fig. 5.22 shows the characteristics of the axial stress through the coil along the radius at midplane and $z = L/2$. Note that due to traction free boundary conditions, axial stress at

Table 5.3 Parameters for the 23 T isotropic coil.

Name	Symbol	Value	Unit
Inner radius	a	100.00	mm
Outer radius	b	136.50	mm
Half length	L	28.00	mm
Elastic modulus	E	111.00	GPa
Poison's ratio	ν	0.30	
Current density	J	530.10	A/mm^2

the top of the coil is zero. The Green's functions solution shows a good correlation with the finite element for axial stress. Fig. 5.23 shows the shear stress at $z = L/2$. The maximum difference between the results of finite element methods and Greens functions solution for the shear stress is about 6%. Shear stress is zero at $z = L$ due to the boundary conditions, and it is zero at midplane because of symmetry. Figs. 5.24 - 5.26 are plots of the normal strains. These graphs show the same close relationship between the Green's functions solution and finite element analysis.

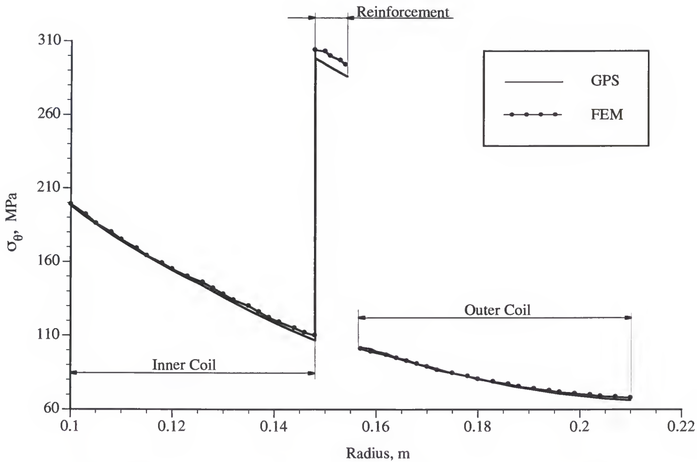

Figure 5.2 Tangential stress comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

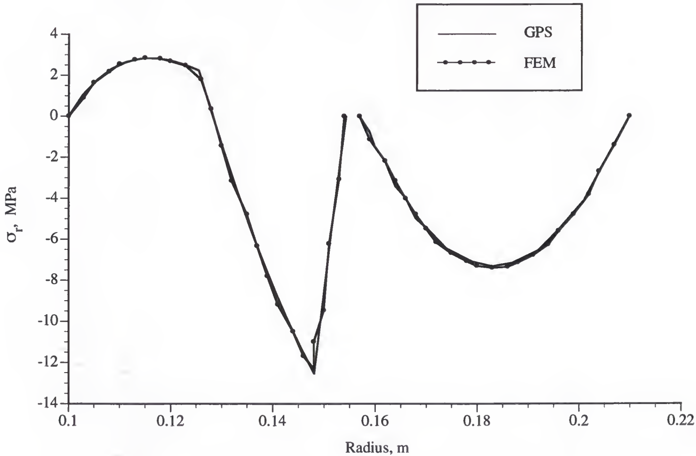

Figure 5.3 Radial stress comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

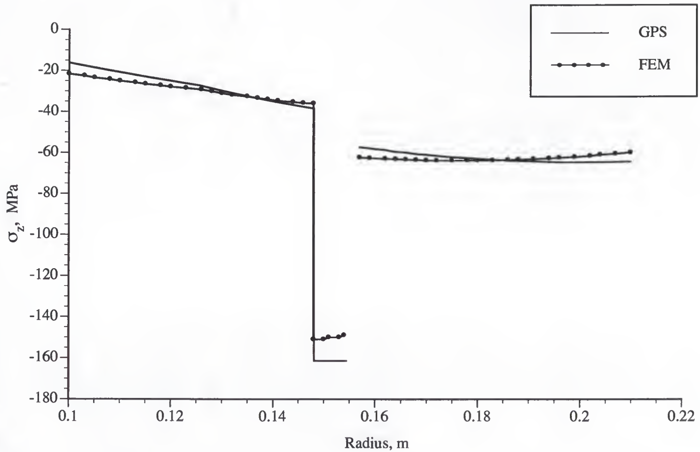

Figure 5.4 Axial stress comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

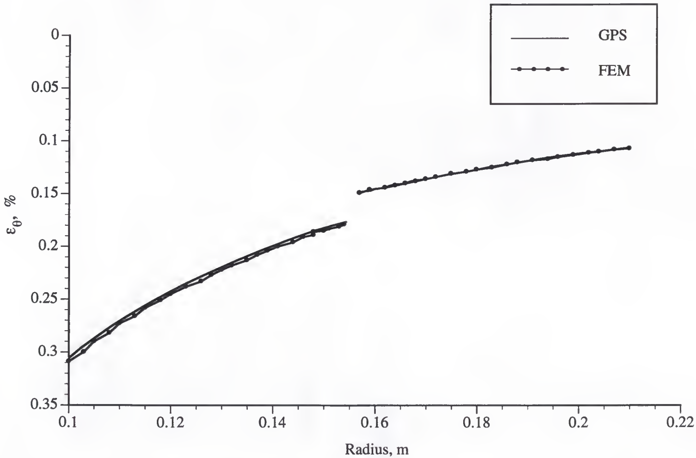

Figure 5.5 Tangential strain comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

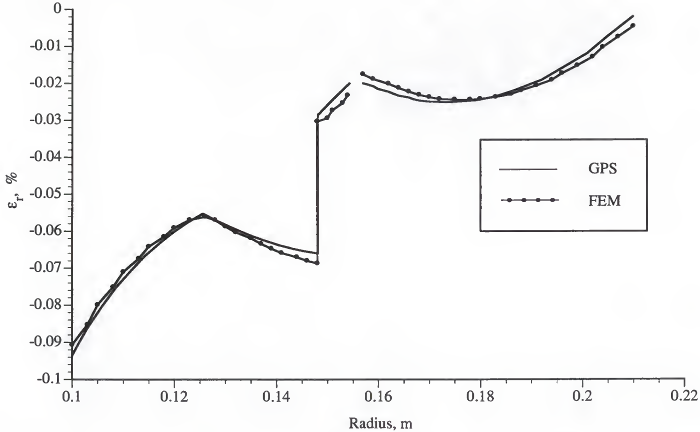

Figure 5.6 Radial Strain comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

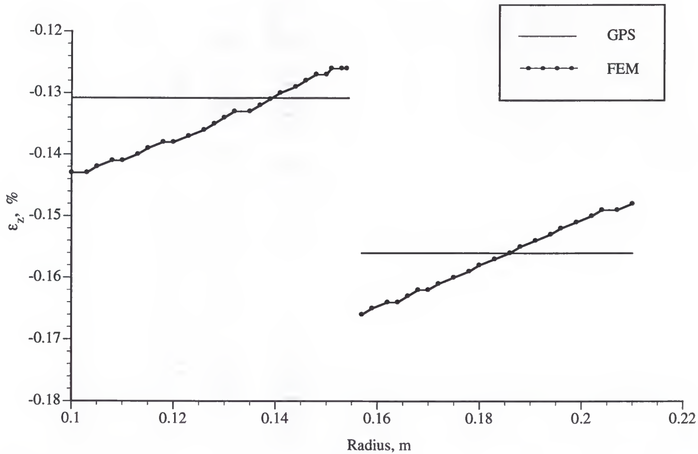

Figure 5.7 Axial Strain comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

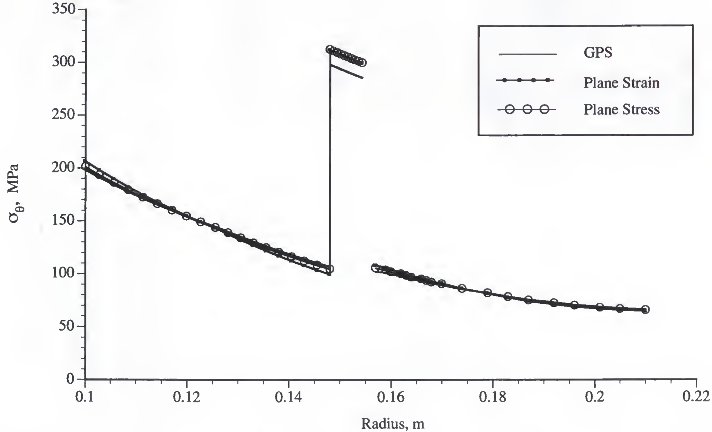

Figure 5.8 Tangential stress comparison between Generalized Plane Strain (GPS) and plane theories of a 15 T superconducting magnet.

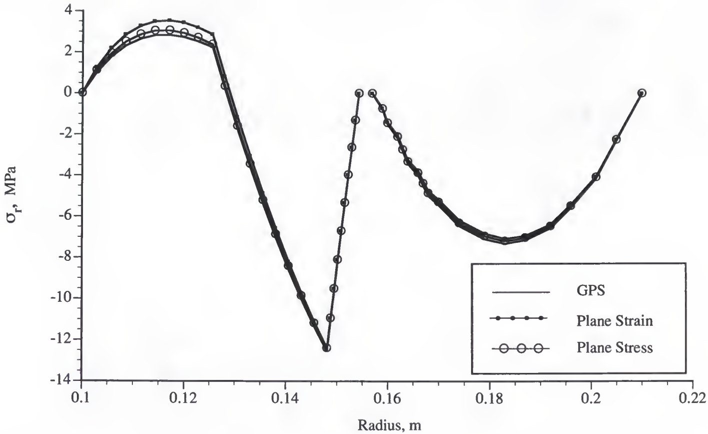

Figure 5.9 Radial stress comparison between Generalized Plane Strain (GPS) and plane theories of a 15 T superconducting magnet.

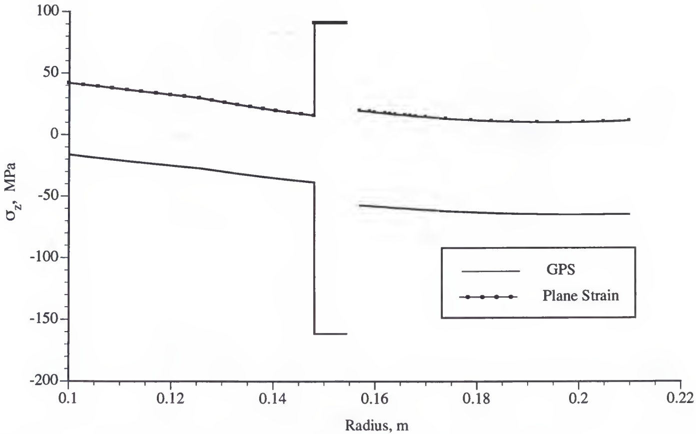

Figure 5.10 Axial stress comparison between Generalized Plane Strain (GPS) and plane strain theory of a 15 T superconducting magnet.

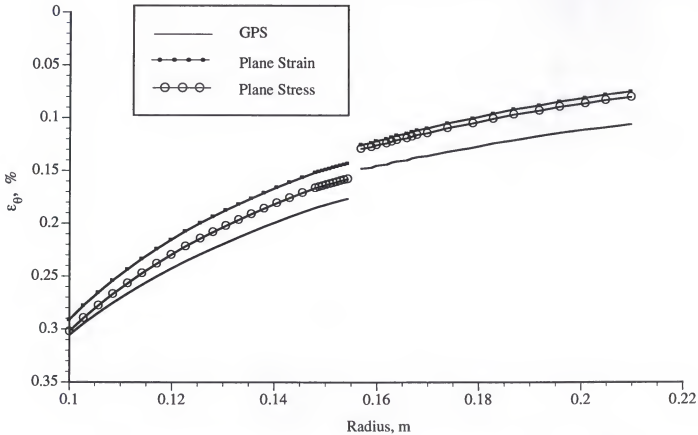

Figure 5.11 Tangential strain comparison between Generalized Plane Strain (GPS) and plane theories of a 15 T superconducting magnet.

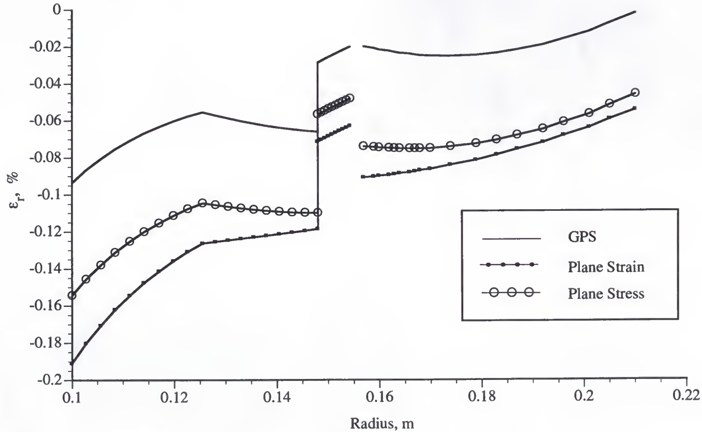

Figure 5.12 Radial Strain comparison between Generalized Plane Strain (GPS) and plane theories of a 15 T superconducting magnet.

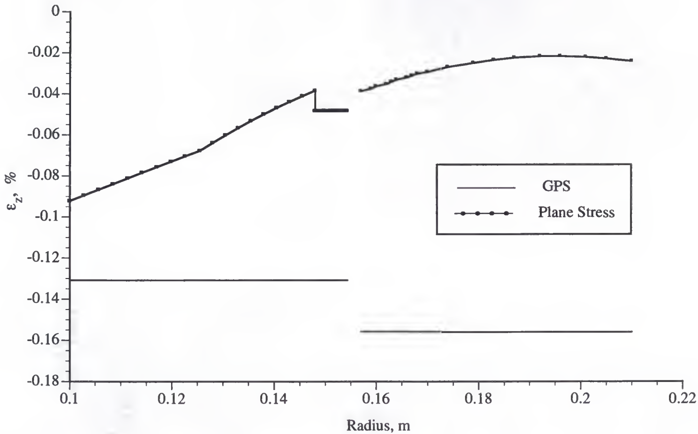

Figure 5.13 Axial Strain comparison between Generalized Plane Strain (GPS) and plane stress theory of a 15 T superconducting magnet.

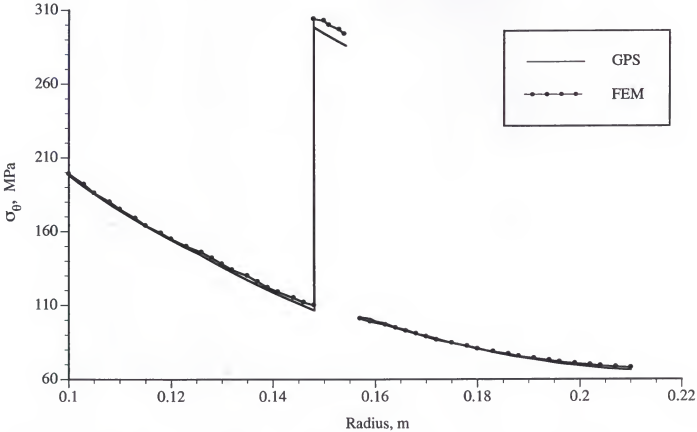

Figure 5.14 Thermal tangential stress comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

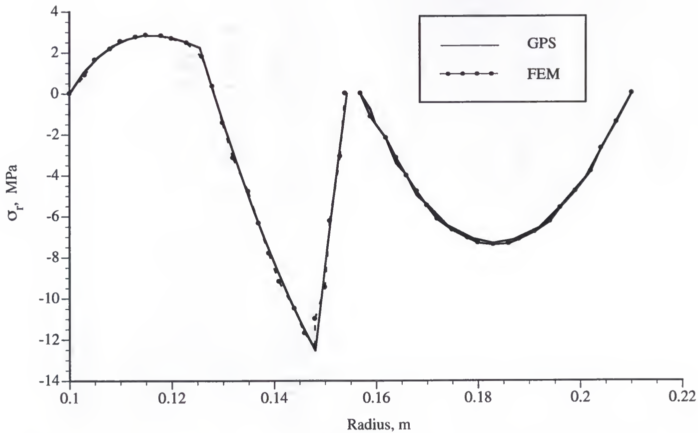

Figure 5.15 Thermal radial stress comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

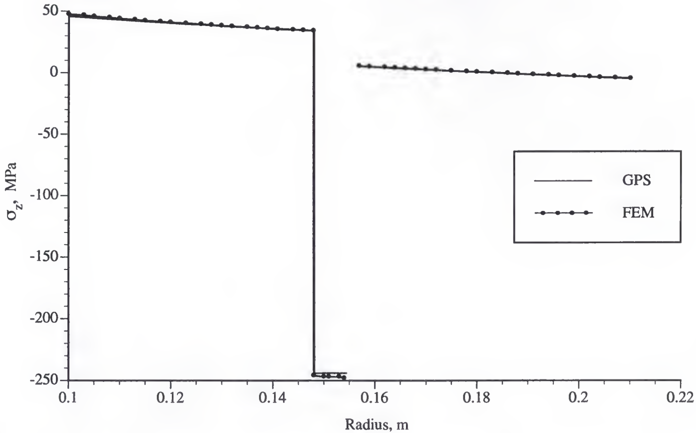

Figure 5.16 Thermal axial stress comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

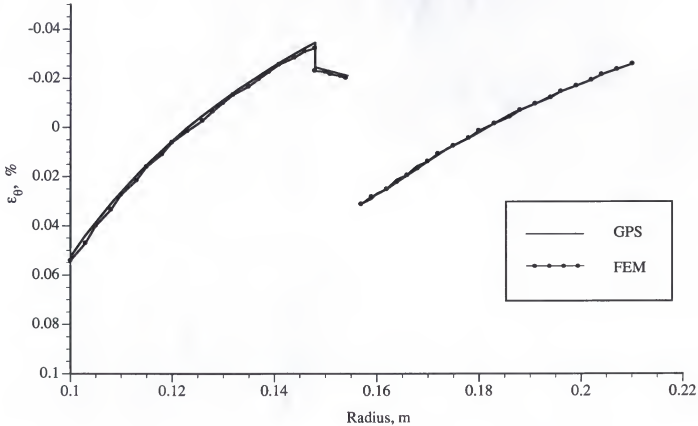

Figure 5.17 Thermal tangential strain comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

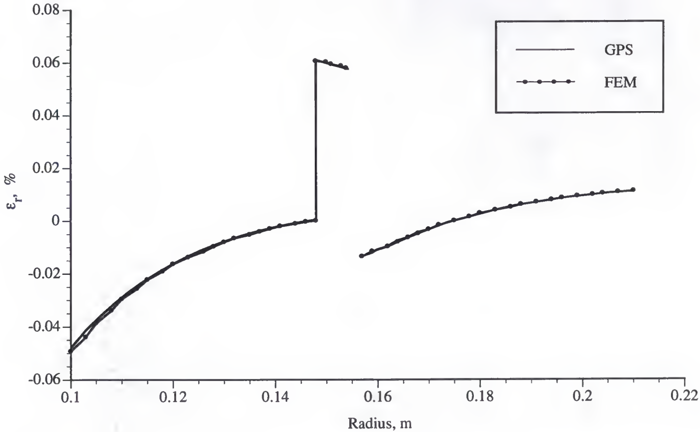

Figure 5.18 Thermal radial Strain comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

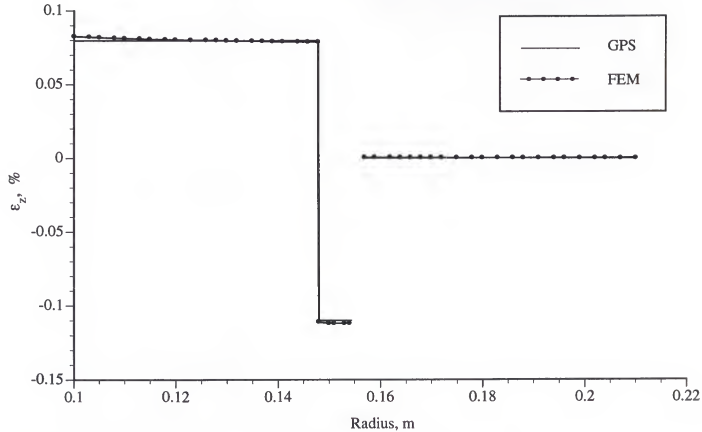

Figure 5.19 Thermal axial Strain comparison between Generalized Plane Strain (GPS) and Finite Element Methods (FEM) of a 15 T superconducting magnet.

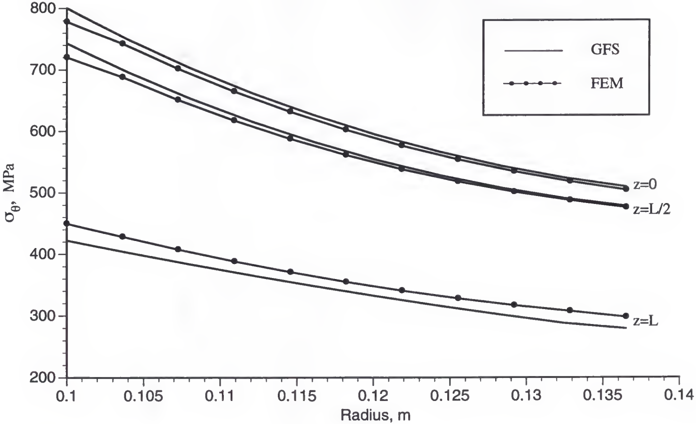

Figure 5.20 Tangential stress comparison between Green's Function Solutions (GFS) and Finite Element Methods (FEM) of a 23 T superconducting magnet.

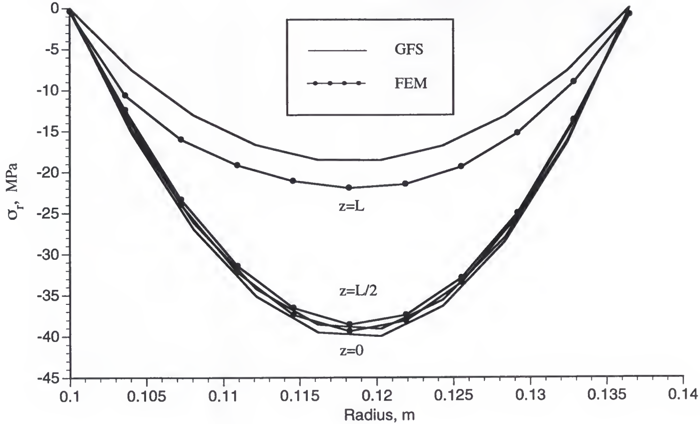

Figure 5.21 Radial stress comparison between Green's Function Solutions (GFS) and Finite Element Methods (FEM) of a 23 T superconducting magnet.

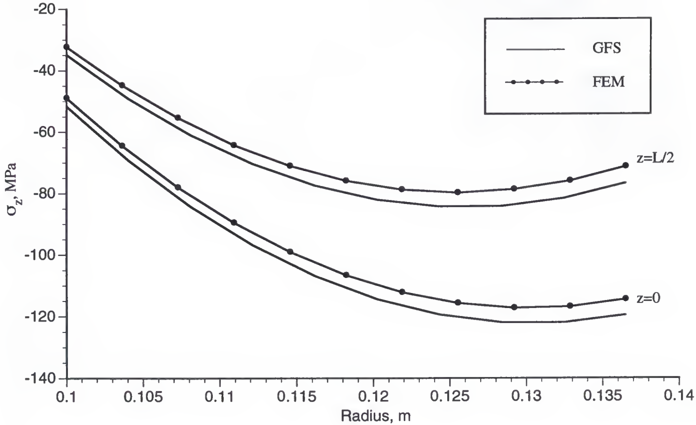

Figure 5.22 Axial stress comparison between Green's Function Solutions (GFS) and Finite Element Methods (FEM) of a 23 T superconducting magnet.

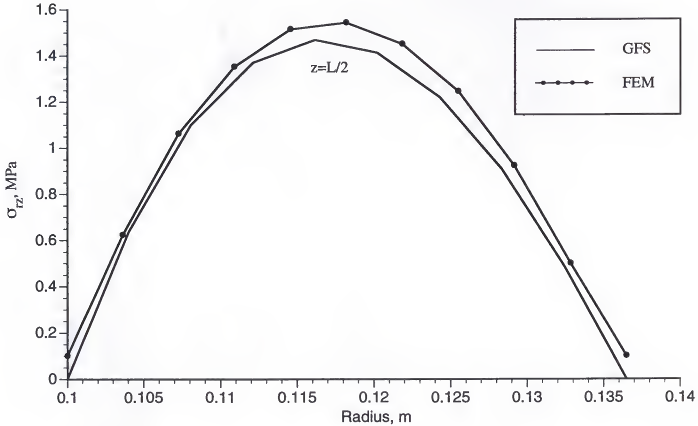

Figure 5.23 Shear stress comparison between Green's Function Solutions (GFS) and Finite Element Methods (FEM) of a 23 T superconducting magnet.

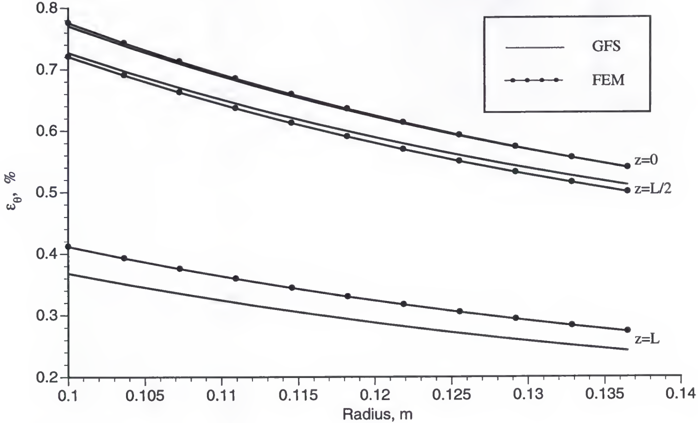

Figure 5.24 Tangential strain comparison between Green's Function Solutions (GFS) and Finite Element Methods (FEM) of a 23 T superconducting magnet.

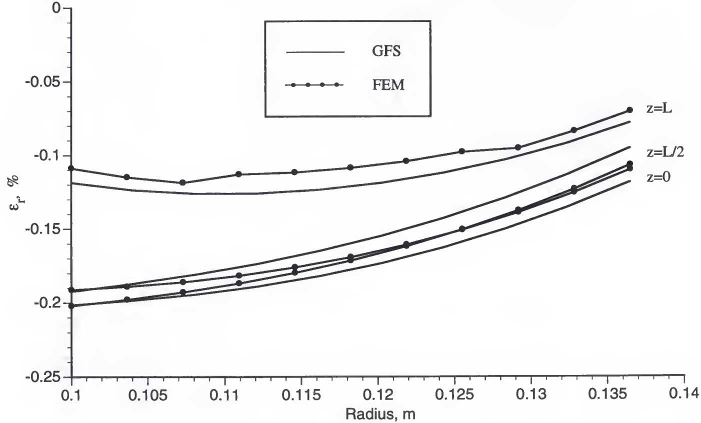

Figure 5.25 Radial strain comparison between Green's Function Solutions (GFS) and Finite Element Methods (FEM) of a 23 T superconducting magnet.

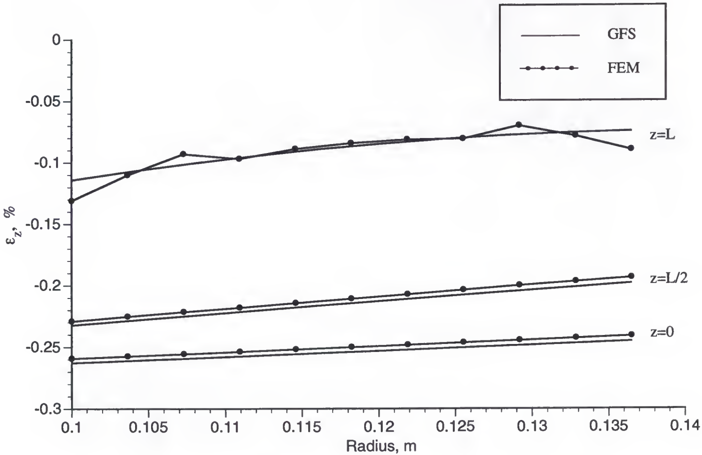

Figure 5.26 Axial strain comparison between Green's Function Solutions (GFS) and Finite Element Methods (FEM) of a 23 T superconducting magnet.

CHAPTER 6
SUMMARY AND CONCLUSIONS

Analytical closed form solutions for the distribution of stress and strain have been developed in high field solenoid magnets, including superconducting magnets. These solutions are presented in the form of generalized plane strain analysis and the Green's functions solution. The generalized plane strain analysis provided an analytical formulation for the magnetic and thermal state of stress at the midplane of a coil, where shear stress is negligible. This formulation considers the effect of the axial body force between the midplane and the end of the coil. The three-dimensional stress analysis in terms of Green's functions provided solutions for components of stress (including the shear stress) throughout the coil. The Green's functions method permits the development of a solution irrespective of the type of the field or its distribution within the coil. Thus, the Green's functions may be used for both thermal and magnetic stresses. The problem was formulated in terms of stress functions. Green's functions were derived by exploiting finite Hankel and finite Fourier transforms. Boundary conditions were satisfied by introducing a complementary solution for the axial stress function. The radial Green's function with the superposition of the complementary and the axial Green's function provided a comprehensive analytical solution for the stresses.

A comparison of the analytical results with the results of finite element methods was presented to confirm the validity of the solutions. The generalized plane strain showed a good agreement with the finite element methods for the radial and tangential stresses and strains. The resulting axial stress and strain from the generalized plane strain analysis were in an acceptable range (less than 9% error) compared to the finite element methods. The generalized plane strain analysis was also compared to other plane theories. The

generalized plane strain improved the prediction of axial stress and strain significantly compared to plane stress and plane strain theories. The resulting stress and strain from the Green's functions solution showed a good correlation with the finite element methods.

The Green's functions solution provided, for the first time, a complete analytical stress solution for an isotropic coil. This solution should be used as a foundation for the stress analysis of multilayer magnets. The future work should also extend this solution for an orthotropic coil. The use of the Green's functions solution is not restricted to magnets. It can also be used for other elasticity problems with physical or fictitious body forces.

APPENDIX A
EQUATIONS FOR THE CONSTANTS OF THE THERMAL STRESS ANALYSIS

In this appendix the axial strain and arbitrary constants of the thermal stress formulation, based upon generalized plane strain analysis, are computed for the different kinds of boundary conditions.

The radial displacement and the radial and axial stresses are expressed by Eqs. (A.1), (A.2) and (A.3), respectively.

$$u = D_1 r^k + D_2 r^{-k} + A_1 \varepsilon_z^{tot} r + A_2 r \tag{A.1}$$

$$\begin{aligned}\sigma_r &= (C_{12} + kC_{22}) D_1 r^{k-1} + (C_{12} - kC_{22}) D_2 r^{-k-1} \\ &+ (C_{12} + C_{22}) A_1 \varepsilon_z^{tot} + (C_{12} + C_{22}) A_2 + C_{23} \varepsilon_z^{tot} \\ &- C_{12} \alpha_\theta \Delta T - C_{22} \alpha_r \Delta T - C_{23} \alpha_z \Delta T\end{aligned} \tag{A.2}$$

$$\begin{aligned}\sigma_z &= (C_{13} + kC_{23}) D_1 r^{k-1} + (C_{13} - kC_{23}) D_2 r^{-k-1} \\ &+ (C_{13} + C_{23}) A_1 \varepsilon_z^{tot} + (C_{13} + C_{23}) A_2 + C_{33} \varepsilon_z^{tot} \\ &- C_{13} \alpha_\theta \Delta T - C_{23} \alpha_r \Delta T - C_{33} \alpha_z \Delta T\end{aligned} \tag{A.3}$$

For a single layer coil with constant current density and uniform material properties, the boundary conditions are

$$\sigma_r = 0 \text{ at } r = a_1 \tag{A.4}$$

$$\sigma_r = 0 \text{ at } r = a_2 \tag{A.5}$$

$$\int_{a_1}^{a_2} 2\pi r \sigma_z dr = 0 . \tag{A.6}$$

Evaluating equation (A.2) at the inner radius (a_1) results in

$$a_{11}D_1 + a_{12}D_2 + a_{13}\varepsilon_z^{tot} = b_1 \tag{A.7}$$

where the constants are given by

$$\begin{aligned} a_{11} &= (C_{12} + kC_{22})a_1^{\,k-1} \\ a_{12} &= (C_{12} - kC_{22})a_1^{\,-k-1} \\ a_{13} &= (C_{12} + C_{22})A_1 + C_{23} \\ b_1 &= -(C_{12} + C_{22})A_2 + C_{12}\alpha_\theta \Delta T + C_{22}\alpha_r \Delta T + C_{23}\alpha_z \Delta T. \end{aligned} \tag{A.8}$$

Evaluating Eq. (A.2) at the outer radius (a_2) results in

$$a_{21}D_1 + a_{22}D_2 + a_{23}\varepsilon_z^{tot} = b_2 \tag{A.9}$$

where the constants are given by

$$\begin{aligned} a_{21} &= (C_{12} + kC_{22})a_2^{\,k-1} \\ a_{22} &= (C_{12} - kC_{22})a_2^{\,-k-1} \\ a_{23} &= a_{13} \\ b_2 &= b_1. \end{aligned} \tag{A.10}$$

Integrating Eq. (A.3) in Eq. (A.6) results in

$$a_{31}D_1 + a_{32}D_2 + a_{33}\varepsilon_z^{tot} = b_3 \tag{A.11}$$

where the constants are given by

$$\begin{aligned} a_{31} &= (C_{13} + kC_{23})\frac{a_2^{\,k+1} - a_1^{\,k+1}}{k+1} \\ a_{32} &= (C_{13} - kC_{23})\frac{a_2^{\,-k+1} - a_1^{\,-k+1}}{-k+1} \\ a_{33} &= \left[(C_{13} + C_{23})A_1 + C_{33}\right]\frac{a_2^{\,2} - a_1^{\,2}}{2} \\ b_3 &= -\left[(C_{13} + C_{23})A_2 - C_{13}\alpha_\theta \Delta T - C_{23}\alpha_r \Delta T - C_{33}\alpha_z \Delta T\right]\frac{a_2^{\,2} - a_1^{\,2}}{2}. \end{aligned} \tag{A.12}$$

The set of linear Eqs. (A.7), (A.9), and (A.11) may be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{A.13}$$

which is solved in the standard way as given by Eq. (A.14).

$$\begin{bmatrix} D_1 \\ D_2 \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{A.14}$$

For a single layer coil with a reinforcement, the boundary conditions are applied to the coil and the reinforcement.

$$\sigma_r^{(1)} = 0 \ \text{ at } \ r = a_1 \tag{A.15}$$

$$\sigma_r^{(1)} = \sigma_r^{(2)} \ \text{ at } \ r = a_2 \tag{A.16}$$

$$u_r^{(1)} = u_r^{(2)} \ \text{ at } \ r = a_2 \tag{A.17}$$

$$\sigma_r^{(2)} = 0 \ \text{ at } \ r = a_3 \tag{A.18}$$

The axial force equilibrium applies to the coil and reinforcement.

$$\int_{a_1}^{a_3} 2\pi r \sigma_z dr = 0 \tag{A.19}$$

Evaluating Eq. (A.2) at the inner radius (a_1) results in

$$a_{11}D_1 + a_{12}D_2 + a_{15}\varepsilon_z^{tot} = b_1 \tag{A.20}$$

where the constants are given by

$$a_{11} = \left(C_{12} + kC_{22}\right)a_1^{\,k-1} \tag{A.21}$$

$$a_{12} = \left(C_{12} - kC_{22}\right)a_1^{\,-k-1}$$

$$a_{15} = \left(C_{12} + C_{22}\right)A_1 + C_{23}$$

$$b_1 = -\left(C_{12} + C_{22}\right)A_2 + C_{12}\alpha_\theta \Delta T + C_{22}\alpha_r \Delta T + C_{23}\alpha_z \Delta T.$$

Evaluating Eq. (A.2) at the interface between the coil and the reinforcement (a_2) results in

$$a_{21}D_1 + a_{22}D_2 + a_{23}D_1' + a_{24}D_2' + a_{25}\varepsilon_z^{tot} = b_2 \tag{A.22}$$

where the constants are given by

$$a_{21} = \left(C_{12} + kC_{22}\right)a_2^{\,k-1} \tag{A.23}$$

$$a_{22} = \left(C_{12} - kC_{22}\right)a_2^{\,-k-1}$$

$$a_{23} = -\left(C_{12}' + k'C_{22}'\right)a_2^{\,k'-1}$$

$$a_{24} = -\left(C_{12}' - k'C_{22}'\right)_2 a_2^{\,-k'-1}$$

$$a_{25} = \left[\left(C_{12} + C_{22}\right)A_1 + C_{23}\right] - \left[\left(C_{12}' + C_{22}'\right)A_1' + C_{23}'\right]$$

$$b_2 = \left[\left(C_{12}' + C_{22}'\right)A_2' - C_{12}'\alpha_\theta' \Delta T - C_{22}'\alpha_r' \Delta T - C_{23}'\alpha_z' \Delta T\right] -$$
$$\left[\left(C_{12} + C_{22}\right)A_2 - C_{12}\alpha_\theta \Delta T - C_{22}\alpha_r \Delta T - C_{23}\alpha_z \Delta T\right],$$

and the unprimed and primed quantities refer to the coil and the reinforcement, respectively. Evaluating Eq. (A.1) for the displacement at the interface results in

$$a_{31}D_1 + a_{32}D_2 + a_{33}D_1' + a_{34}D_2' + a_{35}\varepsilon_z^{tot} = b_3 \tag{A.24}$$

where the constants are given by

$$a_{31} = a_2^{\,k} \qquad a_{32} = a_2^{\,-k} \tag{A.25}$$

$$a_{33} = -a_2^{\,k'} \qquad a_{34} = -a_2^{\,-k'}$$

$$a_{35} = A_1 a_2 - A_1' a_2 \qquad b_3 = A_2' a_2 - A_2 a_2.$$

Evaluating Eq. (A.2) at the outside radius of the reinforcement (a_3) results in

$$a_{43}D_1' + a_{44}D_2' + a_{45}\varepsilon_z^{tot} = b_4 \tag{A.26}$$

where the constants are given by

$$\begin{aligned}
a_{43} &= (C_{12}' + k'C_{22}')a_3^{\,k'-1} \\
a_{44} &= (C_{12}' - k'C_{22}')a_3^{\,-k'-1} \\
a_{45} &= (C_{12}' + C_{22}')A_1' + C_{23}' \\
b_4 &= -(C_{12}' + C_{22}')A_2' + C_{12}'\alpha_\theta'\Delta T + C_{22}'\alpha_r'\Delta T + C_{23}'\alpha_z'\Delta T.
\end{aligned} \tag{A.27}$$

Integrating Eq. (A.3) through coil and reinforcement in Eq. (A.19) results in

$$a_{51}D_1 + a_{52}D_2 + a_{53}D_1' + a_{54}D_2' + a_{55}\varepsilon_z^{tot} = b_5 \tag{A.28}$$

where the constants are given by

$$\begin{aligned}
a_{51} &= (C_{13} + kC_{23})\frac{a_2^{\,k+1} - a_1^{\,k+1}}{k+1} \\
a_{52} &= (C_{13} - kC_{23})\frac{a_2^{\,-k+1} - a_1^{\,-k+1}}{-k+1} \\
a_{53} &= (C_{13}' + k'C_{23}')\frac{a_3^{\,k'+1} - a_2^{\,k'+1}}{k'+1} \\
a_{54} &= (C_{13}' - k'C_{23}')\frac{a_3^{\,-k'+1} - a_2^{\,-k'+1}}{-k'+1} \\
a_{55} &= \left[(C_{13} + C_{23})A_1 + C_{33}\right]\frac{a_2^{\,2} - a_1^{\,2}}{2} + \left[(C_{13}' + C_{23}')A_1 + C_{33}'\right]\frac{a_3^{\,2} - a_2^{\,2}}{2} \\
b_5 &= -\left[(C_{13} + C_{23})A_2 - C_{13}\alpha_\theta\Delta T - C_{23}\alpha_r\Delta T - C_{33}\alpha_z\Delta T\right]\frac{a_2^{\,2} - a_1^{\,2}}{2} \\
&\quad -\left[(C_{13}' + C_{23}')A_2' - C_{13}'\alpha_\theta'\Delta T - C_{23}'\alpha_r'\Delta T - C_{33}'\alpha_z'\Delta T\right]\frac{a_3^{\,2} - a_2^{\,2}}{2}.
\end{aligned} \tag{A.29}$$

The set of linear Eqs. (A.20), (A.22), (A.24), (A.26), and (A.28) may be written in the matrix form

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_1' \\ D_2' \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \quad \text{(A.30)}$$

which is solved as given by Eq. (A.31).

$$\begin{bmatrix} D_1 \\ D_2 \\ D_1' \\ D_2' \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \quad \text{(A.31)}$$

For a general multilayer coil (n layers) the boundary conditions are applied to each layer.

$$\sigma_r^{(1)} = 0 \text{ at } r = a_1 \quad \text{(A.32)}$$

$$\sigma_r^{(j)} = \sigma_r^{(j+1)} \text{ at } r = a_{j+1};\ j = 1, n-1 \quad \text{(A.33)}$$

$$u_r^{(j)} = u_r^{(j+1)} \text{ at } r = a_{j+1};\ j = 1, n-1 \quad \text{(A.34)}$$

$$\sigma_r^{(n)} = 0 \text{ at } r = a_{n+1} \quad \text{(A.35)}$$

The axial force equilibrium also applies to all layers.

$$\int_{a_1}^{a_{n+1}} 2\pi r \sigma_z dr = 0 \quad \text{(A.36)}$$

The derivation of the equations for the constants is the same as the previous cases. The matrix of coefficients for the system of $2n+1$ equations and unknowns is given by Eq. (A.37).

$$\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & \cdots & & & a_{1\,2n+1} \\
a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & \cdots & & & a_{2\,2n+1} \\
a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & \cdots & & & a_{3\,2n+1} \\
0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} & 0 & \cdots & & & a_{4\,2n+1} \\
0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} & 0 & \cdots & & & a_{5\,2n+1} \\
\vdots & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & & & & & 0 & a_{2n\,2n-1} & a_{2n\,2n} & a_{2n\,2n+1} \\
a_{2n+1\,1} & a_{2n+1\,2} & \cdots & & & & & & & a_{2n+1\,2n} & a_{2n+1\,2n+1}
\end{bmatrix} \tag{A.37}$$

The solution to this system of equations is obtained by matrix inversion.

APPENDIX B
PARTIAL DIFFERENTIAL EQUATIONS FOR THE STRESS FUNCTIONS

In this appendix a detailed derivation of the partial differential equations for the stress functions are given.

For an axisymmetric distribution of magnetic body forces **X**(r,z) acting in the interior of an elastic isotropic coil, the equilibrium equation in the vector form is given by Eq. (B.1).

$$\nabla.\boldsymbol{\sigma}+\mathbf{X}=0 \tag{B.1}$$

The constitutive equation relating the stress tensor to the strain tensor is expressed by Eq. (B.2).

$$\boldsymbol{\sigma}=\lambda\, tr(\boldsymbol{\varepsilon})\mathbf{I}+2\mu\boldsymbol{\varepsilon} \tag{B.2}$$

The displacement vector $\mathbf{u}_{(r,z)}$ is related to the strain tensor as given by Eq. (B.3).

$$\boldsymbol{\varepsilon}=\frac{1}{2}\left[\nabla\mathbf{u}+\nabla^{\mathrm{T}}\mathbf{u}\right] \tag{B.3}$$

Incorporating Eq. (B.3) into Eq. (B.2) yields an expression for the stress tensor in terms of displacement vector.

$$\boldsymbol{\sigma}=\lambda(\nabla.\mathbf{u})\mathbf{I}+\mu\left[\nabla\mathbf{u}+\nabla^{\mathrm{T}}\mathbf{u}\right] \tag{B.4}$$

Introducing Eq. (B.4) into Eq. (B.1) gives

$$\lambda\nabla.[(\nabla.\mathbf{u})\mathbf{I}]+\mu\nabla.\left[\nabla\mathbf{u}+\nabla^{\mathrm{T}}\mathbf{u}\right]+\mathbf{X}=0. \tag{B.5}$$

From continuum mechanics,[32] we may write

$$\nabla.[(\nabla.\mathbf{u})\mathbf{I}] = \nabla(\nabla.\mathbf{u})$$
$$\nabla.(\nabla\mathbf{u}) = \nabla(\nabla.\mathbf{u})$$
$$\nabla.(\nabla^{T}\mathbf{u}) = \nabla^{2}\mathbf{u}. \tag{B.6}$$

Incorporating the expressions form Eq. (B.6) into Eq. (B.5) gives

$$(\lambda + \mu)\nabla(\nabla.\mathbf{u}) + \mu\nabla^{2}\mathbf{u} + \mathbf{X} = 0. \tag{B.7}$$

From the Helmholtz theorem, any vector satisfying Eq. (B.7) may be resolved into the sum of a gradient and a curl

$$\mathbf{u} = \nabla\phi + \nabla\times\mathbf{A} \tag{B.8}$$

where $\phi_{(r,z)}$ is a scalar potential and $\mathbf{A}_{(r,z)}$ is a vector potential such that $\nabla.\mathbf{A} = 0$. Incorporating the displacement vector from Eq. (B.8) into Eq. (B.7) gives

$$(\lambda + \mu)\nabla[\nabla.(\nabla\phi + \nabla\times\mathbf{A})] + \mu\nabla^{2}(\nabla\phi + \nabla\times\mathbf{A}) + \mathbf{X} = 0. \tag{B.9}$$

Nevertheless,[32] the divergence of the curl of any vector is zero $\nabla.(\nabla\times\mathbf{A}) = 0$, the divergence of the gradient of a scalar is the Laplacian of a scalar $\nabla.(\nabla\phi) = \nabla^{2}\phi$, the Laplacian of the gradient of a scalar is the gradient of the Laplacian of a scalar $\nabla^{2}(\nabla.\phi) = \nabla.(\nabla^{2}\phi)$ and finally the Laplacian of the curl of a vector is the curl of the Laplacian of a vector $\nabla^{2}(\nabla\times\mathbf{A}) = \nabla\times(\nabla^{2}\mathbf{A})$. By using these identities Eq. (B.9) is reduced to

$$(\lambda + 2\mu)\nabla(\nabla^{2}\phi) + \mu\nabla\times(\nabla^{2}\mathbf{A}) + \mathbf{X} = 0. \tag{B.10}$$

The independent potential functions ϕ and $\mathbf{A}$ may be expressed as

$$\mathbf{A} = \alpha\nabla\times\Psi$$
$$\phi = \beta\nabla.\Psi \tag{B.11}$$

where α and β are arbitrary constants and the vector Ψ is the stress function vector.

Introducing Eq. (B.11) into Eq. (B.10) gives

$$\beta(\lambda+2\mu)\nabla\left[\nabla^2(\nabla.\mathbf{\Psi})\right]+\alpha\mu\nabla\times\left[\nabla^2(\nabla\times\mathbf{\Psi})\right]+\mathbf{X}=0 \tag{B.12}$$

or

$$\beta(\lambda+2\mu)\nabla\left[\nabla.(\nabla^2\mathbf{\Psi})\right]+\alpha\mu\nabla\times\left[\nabla\times(\nabla^2\mathbf{\Psi})\right]+\mathbf{X}=0. \tag{B.13}$$

Substituting $\nabla\left[\nabla.(\nabla^2\mathbf{\Psi})\right]$ in Eq. (B.13) with $\nabla^2(\nabla^2\mathbf{\Psi})+\nabla\times\left[\nabla\times(\nabla^2\mathbf{\Psi})\right]$ yields Eq. (B.14).

$$\beta(\lambda+2\mu)\nabla^4\mathbf{\Psi}+[\beta(\lambda+2\mu)+\mu\alpha]\nabla\times[\nabla\times(\nabla^2\mathbf{\Psi})]+\mathbf{X}=0 \tag{B.14}$$

The second term of Eq. (B.13)can be eliminated by making $[\beta(\lambda+2\mu)+\mu\alpha]$ to be zero. The first term of Eq. (B.13) can be simplified by choosing β as $\frac{1}{\lambda+2\mu}$, which results in α equal to $-\frac{1}{\mu}$. Replacing α and β respectively with $-\frac{1}{\mu}$ and $\frac{1}{\lambda+2\mu}$ in the Eq. (B.14) yields a partial differential equation for the vector $\mathbf{\Psi}$.

$$\nabla^4\mathbf{\Psi}+\mathbf{X}=0 \tag{B.15}$$

Rewriting Eq. (B.15) in component form yields

$$\begin{aligned}
&\left(\nabla^2-\frac{1}{r^2}\right)^2\Psi_r-\frac{4}{r^4}\frac{\partial^2\Psi_r}{\partial\theta^2}-\frac{4}{r^2}\left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial\Psi_\theta}{\partial\theta}+X_r=0 \\
&\left(\nabla^2-\frac{1}{r^2}\right)^2\Psi_\theta-\frac{4}{r^4}\frac{\partial^2\Psi_\theta}{\partial\theta^2}+\frac{4}{r^2}\left(\nabla^2-\frac{1}{r^2}\right)\frac{\partial\Psi_r}{\partial\theta}+X_\theta=0 \\
&\nabla^4\Psi_z+X_z=0.
\end{aligned} \tag{B.16}$$

Eq. (B.16) defines the three partial differential equations of stress functions.

APPENDIX C
ORTHOGONALITY PROPERTY OF THE FOURIER BESSEL KERNEL

In this appendix the orthogonality property of the Fourier Bessel kernel $K_{n(\zeta_i, r)}$ is proved and the expression for the $\|K_{n(\zeta_i, r)}\|$ is obtained.

In cylindrical coordinates (r, θ, z), the Laplace equation $\nabla^2\phi = 0$ that is satisfied by a potential function ϕ is defined by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{C.1}$$

Replacing ϕ by $u_{(r)}e^{-\kappa z}\cos n\theta$, where κ is a real positive quantity and n is an integer number, results in an ordinary differential equation for $u_{(r)}$.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+\left(\kappa^2-\frac{n^2}{r^2}\right)u=0 \tag{C.2}$$

Eq. (C.2) is a Bessel equation of order n. Solution to this equation may be written in terms of Bessel functions of first and second kind.

$$u_{(r)} = AJ_{n(\kappa r)} + BY_{n(\kappa r)} \tag{C.3}$$

Hence, the final solution to ϕ is obtained.

$$\phi_{(r,\theta,z)} = e^{-\kappa z}\cos n\theta\left[AJ_{n(\kappa r)} + BY_{n(\kappa r)}\right] \tag{C.4}$$

Similarly a solution to the Laplace equation for a potential function ψ may be written as

$$\psi_{(r,\theta,z)} = e^{-\lambda z}\cos n\theta\left[\tilde{A}J_{n(\lambda r)} + \tilde{B}Y_{n(\lambda r)}\right] \tag{C.5}$$

where λ is a real positive quantity.

Green's theorem is expressed by following

$$\iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)dV = \iint_S (\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n})dS \tag{C.6}$$

where $\frac{\partial\phi}{\partial n}$ is the directional derivative of ϕ in the direction of the outer normal **n** to the surface element *dS*. The left hand side of Eq. (C.6) vanishes, since both ϕ and ψ satisfy the Laplace equation. Thus, Eq. (C.6) is reduced to

$$\iint_S \psi\frac{\partial\phi}{\partial n}dS = \iint_S \phi\frac{\partial\psi}{\partial n}dS \tag{C.7}$$

where the surface integration is being taken over the surface bounded by a hollow cylinder $a \le r \le b$ and the planes $z = 0$ and $z = +\infty$.

For $z = 0$, the normal to the surface element is expressed by $\mathbf{n} = -z\mathbf{e}_z$. Consequently $\frac{\partial\phi}{\partial n}$ at $z = 0$ reduces to $\kappa\cos n\theta[AJ_{n(\kappa r)} + BY_{n(\kappa r)}]$. Also at $z = 0$, ψ is reduced to $\cos n\theta[\tilde{A}J_{n(\lambda r)} + \tilde{B}Y_{n(\lambda r)}]$. The substitution of ψ and $\frac{\partial\phi}{\partial n}$ into the left hand side of Eq. (C.7), for the flat end part of the surface integral, yields

$$\int_0^{2\pi}\int_a^b \kappa\cos^2 n\theta[AJ_{n(\kappa r)} + BY_{n(\kappa r)}][\tilde{A}J_{n(\lambda r)} + \tilde{B}Y_{n(\lambda r)}]rdrd\theta \tag{C.8}$$

$$= \pi\kappa\int_a^b [AJ_{n(\kappa r)} + BY_{n(\kappa r)}][\tilde{A}J_{n(\lambda r)} + \tilde{B}Y_{n(\lambda r)}]rdr.$$

When $z = +\infty$, $\psi\frac{\partial\phi}{\partial n}$ vanishes, and nothing is contributed to the surface integral.

For the curved surface at the inner radius $r = a$ the normal to the surface element is $\mathbf{n} = -r\mathbf{e}_r$ and consequently $\frac{\partial\phi}{\partial n}$ and ψ may be written as $-\kappa e^{-\kappa z}\cos n\theta[AJ'_{n(\kappa a)} + BY'_{n(\kappa a)}]$ and $e^{-\lambda z}\cos n\theta[\tilde{A}J_{n(\lambda a)} + \tilde{B}Y_{n(\lambda a)}]$, respectively. Hence the corresponding part of the surface integral is defined by Eq. (C.9).

$$\int_0^{2\pi}\int_0^{\infty} -\kappa\cos^2 n\theta e^{-(\kappa+\lambda)z}\left[AJ'_{n(\kappa a)}+BY'_{n(\kappa a)}\right]\left[\tilde{A}J_{n(\lambda a)}+\tilde{B}Y_{n(\lambda a)}\right]adzd\theta \tag{C.9}$$

$$=-\frac{\pi\kappa a}{\lambda+\kappa}\left[AJ'_{n(\kappa a)}+BY'_{n(\kappa a)}\right]\left[\tilde{A}J_{n(\lambda a)}+\tilde{B}Y_{n(\lambda a)}\right]$$

For the curved surface at the outer radius $r = b$ the normal to the surface element is $\mathbf{n} = r\mathbf{e}_r$ and hence, the surface integral corresponding to $r = b$ becomes

$$\int_0^{2\pi}\int_0^{\infty} \kappa\cos^2 n\theta e^{-(\kappa+\lambda)z}\left[AJ'_{n(\kappa b)}+BY'_{n(\kappa b)}\right]\left[\tilde{A}J_{n(\lambda b)}+\tilde{B}Y_{n(\lambda b)}\right]bdzd\theta \tag{C.10}$$

$$=\frac{\pi\kappa b}{\lambda+\kappa}\left[AJ'_{n(\kappa b)}+BY'_{n(\kappa b)}\right]\left[\tilde{A}J_{n(\lambda b)}+\tilde{B}Y_{n(\lambda b)}\right].$$

Working out the other side of Eq. (C.7) in a similar way, and adding different parts of the surface integral together, yields

$$(\kappa-\lambda)\int_a^b\left[AJ_{n(\kappa r)}+BY_{n(\kappa r)}\right]\left[\tilde{A}J_{n(\lambda r)}+\tilde{B}Y_{n(\lambda r)}\right]rdr \tag{C.11}$$

$$\begin{aligned}=\frac{1}{\lambda+\kappa}\Big\{&\kappa a\left[AJ'_{n(\kappa a)}+BY'_{n(\kappa a)}\right]\left[\tilde{A}J_{n(\lambda a)}+\tilde{B}Y_{n(\lambda a)}\right]\\&-\kappa b\left[AJ'_{n(\kappa b)}+BY'_{n(\kappa b)}\right]\left[\tilde{A}J_{n(\lambda b)}+\tilde{B}Y_{n(\lambda b)}\right]\\&-\lambda a\left[AJ_{n(\kappa a)}+BY_{n(\kappa a)}\right]\left[\tilde{A}J'_{n(\lambda a)}+\tilde{B}Y'_{n(\lambda a)}\right]\\&+\lambda b\left[AJ_{n(\kappa b)}+BY_{n(\kappa b)}\right]\left[\tilde{A}J'_{n(\lambda b)}+\tilde{B}Y'_{n(\lambda b)}\right]\Big\}.\end{aligned}$$

Thus, if κ and λ are different, then we have

$$\int_a^b\left[AJ_{n(\kappa r)}+BY_{n(\kappa r)}\right]\left[\tilde{A}J_{n(\lambda r)}+\tilde{B}Y_{n(\lambda r)}\right]rdr=0 \tag{C.12}$$

provided that ϕ and ψ satisfy the homogeneous Dirichlet boundary conditions for r in the interval $a \le r \le b$.

$$\left[\tilde{A}J_{n(\lambda a)}+\tilde{B}Y_{n(\lambda a)}\right]=0 \qquad \left[\tilde{A}J_{n(\lambda b)}+\tilde{B}Y_{n(\lambda b)}\right]=0 \tag{C.13}$$

$$\left[AJ_{n(\kappa a)}+BY_{n(\kappa a)}\right]=0 \qquad \left[AJ_{n(\kappa b)}+BY_{n(\kappa b)}\right]=0$$

From conditions in Eq. (C.13) expressions for A, B, $\tilde{A}$ and $\tilde{B}$ are obtained

$$\tilde{A}=Y_{n(\lambda b)} \qquad \tilde{B}=-J_{n(\lambda b)} \tag{C.14}$$

$$A=Y_{n(\kappa b)} \qquad B=-J_{n(\kappa b)}$$

where κ and λ are different roots of the following transcendental equation.

$$J_{n(\zeta_i a)}Y_{n(\zeta_i b)}-J_{n(\zeta_i b)}Y_{n(\zeta_i a)}=0 \tag{C.15}$$

Incorporating the expressions for the constants in the Eq. (C.12) yields

$$\int_a^b\left[J_{n(\kappa r)}Y_{n(\kappa b)}-J_{n(\kappa b)}Y_{n(\kappa r)}\right]\left[J_{n(\lambda r)}Y_{n(\lambda b)}-J_{n(\lambda b)}Y_{n(\lambda r)}\right]rdr=0. \tag{C.16}$$

A Fourier Bessel kernel based upon solution of the Bessel equation with the homogeneous Dirichlet conditions in a finite interval $[a,b]$ is defined by

$$K_{n(\zeta_i,r)}=\left[J_{n(\zeta_i r)}Y_{n(\zeta_i b)}-J_{n(\zeta_i b)}Y_{n(\zeta_i r)}\right] \tag{C.17}$$

where ζ_i is a root of the transcendental equation given by Eq. (C.15). Introducing the definition of $K_{n(\zeta_i,r)}$ into Eq. (C.16) gives the orthogonality property of the Fourier Bessel kernel.

$$\int_a^b rK_{n(\zeta_i,r)}K_{n(\zeta_j,r)}dr=0 \qquad \text{if} \qquad \zeta_i\neq\zeta_j \tag{C.18}$$

To obtain an expression for the norm of the Fourier Bessel kernel, we have to equate κ and λ in the Eq. (C.11). But, this will induce both sides of Eq. (C.11) to vanish. However, we may prevent this from happening by replacing κ with $\lambda+h$ and then by taking the limit as h decreases indefinitely. Thus, Eq. (C.11) may be written as

$$\int_a^b \left[J_{n(\lambda r)}Y_{n(\lambda b)} - J_{n(\lambda b)}Y_{n(\lambda r)}\right]^2 rdr = \lim_{h\to 0}\left\langle \frac{1}{h}\frac{1}{2\lambda+h}\right. \tag{C.19}$$

$$\left\{(\lambda+h)a\left[Y_{n(\lambda b)}J'_{n[(\lambda+h)a]} - J_{n(\lambda b)}Y'_{n[(\lambda+h)a]}\right]\left[Y_{n(\lambda b)}J_{n(\lambda a)} - J_{n(\lambda b)}Y_{n(\lambda a)}\right]\right.$$

$$-(\lambda+h)b\left[Y_{n(\lambda b)}J'_{n[(\lambda+h)b]} - J_{n(\lambda b)}Y'_{n[(\lambda+h)b]}\right]\left[Y_{n(\lambda b)}J_{n(\lambda b)} - J_{n(\lambda b)}Y_{n(\lambda b)}\right]$$

$$-\lambda a\left[Y_{n(\lambda b)}J_{n[(\lambda+h)a]} - J_{n(\lambda b)}Y_{n[(\lambda+h)a]}\right]\left[Y_{n(\lambda b)}J'_{n(\lambda a)} - J_{n(\lambda b)}Y'_{n(\lambda a)}\right]$$

$$\left.\left.+\lambda b\left[Y_{n(\lambda b)}J_{n[(\lambda+h)b]} - J_{n(\lambda b)}Y_{n[(\lambda+h)b]}\right]\left[Y_{n(\lambda b)}J'_{n(\lambda b)} - J_{n(\lambda b)}Y'_{n(\lambda b)}\right]\right\}\right\rangle$$

where A and B have the same expressions as $\tilde{A}$ and $\tilde{B}$ respectively, and are replaced by following.

$$A = \tilde{A} = Y_{n(\lambda b)} \tag{C.20}$$

$$B = \tilde{B} = -J_{n(\lambda b)}$$

Expanding $J_{n[(\lambda+h)r]}$ and $Y_{n[(\lambda+h)r]}$ using Taylor's theorem yields Eq. (C.21).

$$J_{n[(\lambda+h)r]} = J_{n(\lambda r)} + hrJ'_{n(\lambda r)} + O_{(h^2)} \tag{C.21}$$

$$Y_{n[(\lambda+h)r]} = Y_{n(\lambda r)} + hrY'_{n(\lambda r)} + O_{(h^2)}$$

Incorporating Eq. (C.21) into Eq. (C.19) and taking the limit as h goes to zero, results in

$$\int_a^b \left[J_{n(\lambda r)}Y_{n(\lambda b)} - J_{n(\lambda b)}Y_{n(\lambda r)}\right]^2 rdr \tag{C.22}$$

$$=\frac{1}{2}\left\{a^2\left[Y_{n(\lambda b)}J''_{n(\lambda a)} - J_{n(\lambda b)}Y''_{n(\lambda a)}\right]\left[Y_{n(\lambda b)}J_{n(\lambda a)} - J_{n(\lambda b)}Y_{n(\lambda a)}\right]\right.$$

$$+b^2\left[Y_{n(\lambda b)}J'_{n(\lambda b)} - J_{n(\lambda b)}Y'_{n(\lambda b)}\right]^2 - a^2\left[Y_{n(\lambda b)}J'_{n(\lambda a)} - J_{n(\lambda b)}Y'_{n(\lambda a)}\right]^2$$

$$\left.+\frac{a}{\lambda}\left[Y_{n(\lambda b)}J_{n(\lambda a)} - J_{n(\lambda b)}Y_{n(\lambda a)}\right]\left[Y_{n(\lambda b)}J'_{n(\lambda a)} - J_{n(\lambda b)}Y'_{n(\lambda a)}\right]\right\}.$$

Since λ is a root of the Eq. (C.15), we may write

$$J_{n(\lambda a)}Y_{n(\lambda b)} - J_{n(\lambda b)}Y_{n(\lambda a)} = 0. \tag{C.23}$$

Incorporating Eq. (C.23) into Eq. (C.22) yields Eq. (C.24).

$$\int_a^b \left[J_{n(\lambda r)}Y_{n(\lambda b)} - J_{n(\lambda b)}Y_{n(\lambda r)}\right]^2 rdr \tag{C.24}$$

$$= \frac{1}{2}\left\{b^2\left[Y_{n(\lambda b)}J'_{n(\lambda b)} - J_{n(\lambda b)}Y'_{n(\lambda b)}\right]^2 - a^2\left[Y_{n(\lambda b)}J'_{n(\lambda a)} - J_{n(\lambda b)}Y'_{n(\lambda a)}\right]^2\right\}$$

Eq. (C.24) may be simplified furthermore by manipulating the terms in the brackets of the right hand side of the equation. If $y_{1(r)}$ and $y_{2(r)}$ are two independent solutions of $y'' + y'f_{(r)} + yg_{(r)} = 0$, then both satisfy the differential equation.

$$y_1'' + y_1'f_{(r)} + y_1g_{(r)} = 0 \tag{C.25}$$

$$y_2'' + y_2'f_{(r)} + y_2g_{(r)} = 0 \tag{C.26}$$

Eliminating $g_{(r)}$ in Eqs. (C.25) and (C.26) yields

$$\left(y_2y_1'' - y_1y_2''\right) + \left(y_2y_1' - y_1y_2'\right)f_{(r)} = 0. \tag{C.27}$$

The first parenthesis in Eq. (C.27) is the derivative of the second, so that if the latter is denoted by z, Eq. (C.27) is reduced to $z' + zf_{(r)} = 0$. Solution to this differential equation may be written in the form

$$z = y_2y_1' - y_1y_2' = Ae^{-\int f_{(r)}dr} \tag{C.28}$$

where A is an arbitrary constant.

In the case of the Bessel equation, $y'' + \frac{1}{r}y' + \left(\lambda^2 - \frac{n^2}{r^2}\right)y = 0$, we have $f_{(r)} = \frac{1}{r}$, $y_1 = J_{n(\lambda r)}$ and $y_2 = Y_{n(\lambda r)}$. Thus, Eq. (C.28) becomes

$$\lambda\left[Y_{n(\lambda r)}J'_{n(\lambda r)} - Y'_{n(\lambda r)}J_{n(\lambda r)}\right] = Ae^{-\ln r} \tag{C.29}$$

or

$$Y_{n(\lambda r)}J'_{n(\lambda r)} - Y'_{n(\lambda r)}J_{n(\lambda r)} = \frac{A}{\lambda r}. \tag{C.30}$$

Weber's definition for the Bessel function of the second kind $Y_{n}(\lambda r)$ is given by Eq. (C.31).

$$Y_{n}(\lambda r) = \frac{\cos n\pi J_{n}(\lambda r) - J_{-n}(\lambda r)}{\sin n\pi} \tag{C.31}$$

Substituting Weber's definition of $Y_{n}(\lambda r)$, Eq. (C.31), into Eq. (C.30) gives

$$\frac{-1}{\sin n\pi}\left[J'_{n}(\lambda r)J_{-n}(\lambda r) - J_{n}(\lambda r)J'_{-n}(\lambda r)\right] = \frac{A}{\lambda r}. \tag{C.32}$$

An expression for the constant A is obtained by substituting the leading term of the series for Bessel function of the first kind $J_{n}(\lambda r) = \sum_{m=0}^{\infty} \frac{(-1)^{m}(\lambda r/2)^{2m+n}}{m!\Gamma(m+n+1)}$ into the left hand side of Eq. (C.32).

$$\frac{-1}{\sin n\pi}\left[\frac{n}{\Gamma(n+1)}\frac{1}{\Gamma(-n+1)} - \frac{1}{\Gamma(n+1)}\frac{-n}{\Gamma(-n+1)}\right]\frac{1}{\lambda r} = \frac{A}{\lambda r} \tag{C.33}$$

Since $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin n\pi}$, Eq. (C.33) reduces to $A = -\frac{2}{\pi}$. Hence, Eq. (C.30) may be written in the form

$$Y_{n}(\lambda r)J'_{n}(\lambda r) - Y'_{n}(\lambda r)J_{n}(\lambda r) = \frac{-2}{\pi\lambda r}. \tag{C.34}$$

Evaluating Eq. (C.34) at $r = b$ results in

$$Y_{n}(\lambda b)J'_{n}(\lambda b) - Y'_{n}(\lambda b)J_{n}(\lambda b) = \frac{-2}{\pi\lambda b}. \tag{C.35}$$

Evaluating Eq. (C.34) at $r = a$ and using Eq. (C.23) yields Eq. (C.36).

$$Y_{n}(\lambda b)J'_{n}(\lambda a) - Y'_{n}(\lambda a)J_{n}(\lambda b) = \frac{-2}{\pi\lambda a}\frac{J_{n}(\lambda b)}{J_{n}(\lambda a)} \tag{C.36}$$

We may rewrite Eqs. (C.35) and (C.36) in terms of $K_{n}(\zeta_{i}, r)$.

$$\left.\frac{\partial K_{n(\zeta_i,r)}}{\partial r}\right|_{r=a} = -\frac{2}{\pi a}\frac{J_{n(\zeta_i b)}}{J_{n(\zeta_i a)}} \qquad \left.\frac{\partial K_{n(\zeta_i,r)}}{\partial r}\right|_{r=b} = -\frac{2}{\pi b} \tag{C.37}$$

Incorporating the expressions from Eqs. (C.35) and (C.36) into Eq. (C.24) gives

$$\int_a^b \left[J_{n(\lambda r)}Y_{n(\lambda b)} - J_{n(\lambda b)}Y_{n(\lambda r)}\right]^2 rdr = \frac{2}{\pi^2\lambda^2}\left[1 - \frac{J_{n(\lambda b)}^2}{J_{n(\lambda a)}^2}\right]. \tag{C.38}$$

By rewriting the left hand side of Eq. (C.38) as $\left\|K_{n(\zeta_i,r)}\right\|$, the expression for the norm of the Fourier Bessel kernel is obtained from Eq. (C.39).

$$\left\|K_{n(\zeta_i,r)}\right\|^2 = \frac{2}{\pi^2\zeta_i^2}\left[1 - \frac{J_{n(\zeta_i b)}^2}{J_{n(\zeta_i a)}^2}\right] \tag{C.39}$$

APPENDIX D
STRESSES IN TERM OF STRESS FUNCTIONS

In this appendix stresses are derived from the axial and radial stress functions. The displacement vector is defined in terms of stress functions by Eq. (D.1).

$$\mathbf{u} = -\alpha\nabla^2(\Psi_{r(r,z)}\mathbf{e_r} + \Psi_{z(r,z)}\mathbf{e_z}) + (\alpha+\beta)\nabla[\nabla.(\Psi_{r(r,z)}\mathbf{e_r} + \Psi_{z(r,z)}\mathbf{e_z}] \tag{D.1}$$

Rewriting Eq. (D.1) in the component form yields

$$u_r = -\alpha\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + (\alpha+\beta)\frac{\partial\varphi}{\partial r} \tag{D.2}$$

$$u_z = -\alpha\nabla^2\Psi_z + (\alpha+\beta)\frac{\partial\varphi}{\partial z}$$

where φ is given by Eq. (D.3).

$$\varphi = \frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r) + \frac{\partial\Psi_z}{\partial z} \tag{D.3}$$

From the constitutive equation and strain-displacement relationship, the stress tensor may be written in the form of displacement vector.

$$\boldsymbol{\sigma} = \lambda(\nabla.\mathbf{u})\mathbf{I} + \mu\left[\nabla\mathbf{u} + \nabla^{\mathrm{T}}\mathbf{u}\right] \tag{D.4}$$

By using Eq. (D.4) stresses are obtained in terms of displacements.

$$\sigma_r = \lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] + 2\mu\frac{\partial u_r}{\partial r} \tag{D.5}$$

$$\sigma_\theta = \lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] + 2\mu\frac{u_r}{r} \tag{D.6}$$

$$\sigma_z = \lambda\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] + 2\mu\frac{\partial u_z}{\partial z} \tag{D.7}$$

$$\sigma_{zr} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{D.8}$$

Introducing Eq. (D.2) into Eqs. (D.5) through (D.8) gives,

$$\begin{aligned}\sigma_r = &-\lambda\,\alpha\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\right] + \lambda(\alpha+\beta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)\\ &-\lambda\,\alpha\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right) + \lambda\,(\alpha+\beta)\frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z}\right) - 2\mu\alpha\frac{\partial}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\\ &+2\mu(\alpha+\beta)\frac{\partial}{\partial r}\left(\frac{\partial \varphi}{\partial r}\right)\end{aligned} \tag{D.9}$$

$$\begin{aligned}\sigma_\theta = &-\lambda\,\alpha\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\right] + \lambda(\alpha+\beta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)\\ &-\lambda\,\alpha\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right) + \lambda\,(\alpha+\beta)\frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z}\right) - 2\mu\alpha\frac{1}{r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\\ &+2\mu(\alpha+\beta)\frac{1}{r}\left(\frac{\partial \varphi}{\partial r}\right)\end{aligned} \tag{D.10}$$

$$\begin{aligned}\sigma_z = &-\lambda\,\alpha\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\right] + \lambda(\alpha+\beta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)\\ &-\lambda\,\alpha\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right) + \lambda\,(\alpha+\beta)\frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z}\right) - 2\mu\alpha\frac{\partial}{\partial z}\nabla^2\Psi_z\\ &+2\mu(\alpha+\beta)\frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z}\right)\end{aligned} \tag{D.11}$$

$$\sigma_{rz} = \mu\left\{-\alpha\frac{\partial}{\partial z}\left[\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\right] - \alpha\frac{\partial}{\partial r}\nabla^2\Psi_z + 2(\alpha+\beta)\frac{\partial^2\varphi}{\partial r\partial z}\right\}. \tag{D.12}$$

The terms $\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r\right]$ and $\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right)$ may be written as $\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r)\right]$ and $\nabla^2\left(\frac{\partial}{\partial z}\Psi_z\right)$, respectively. Hence, substitution of these terms into Eqs. (D.9)-(D.11) leads to

$$\sigma_r = -\lambda\,\alpha\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r) + \left(\frac{\partial\Psi_z}{\partial z}\right)\right] + \lambda(\alpha+\beta)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2}\right] \tag{D.13}$$
$$-2\mu\alpha\frac{\partial}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + 2\mu(\alpha+\beta)\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right)$$

$$\sigma_\theta = -\lambda\,\alpha\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r)\right] + \lambda(\alpha+\beta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) - \lambda\,\alpha\nabla^2\left(\frac{\partial}{\partial z}\Psi_z\right) \tag{D.14}$$
$$+\lambda(\alpha+\beta)\frac{\partial^2\varphi}{\partial z^2} - 2\mu\alpha\frac{1}{r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + 2\mu(\alpha+\beta)\frac{1}{r}\left(\frac{\partial\varphi}{\partial r}\right)$$

$$\sigma_z = -\lambda\,\alpha\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r)\right] + \lambda(\alpha+\beta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) - \lambda\,\alpha\nabla^2\left(\frac{\partial}{\partial z}\Psi_z\right) \tag{D.15}$$
$$+\lambda(\alpha+\beta)\frac{\partial^2\varphi}{\partial z^2} - 2\mu\alpha\nabla^2\left(\frac{\partial}{\partial z}\Psi_z\right) + 2\mu(\alpha+\beta)\frac{\partial^2\varphi}{\partial z^2},$$

but $\frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r) + \frac{\partial\Psi_z}{\partial z}$ is φ and $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2}$ is the Laplacian of φ. Thus, Eq. (D.13)-(D.15) are reduced to

$$\sigma_r = \lambda\,\beta\nabla^2\varphi - 2\mu\alpha\frac{\partial}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + 2\mu(\alpha+\beta)\frac{\partial^2\varphi}{\partial r^2} \tag{D.16}$$

$$\sigma_\theta = \lambda\,\beta\nabla^2\varphi - 2\mu\alpha\frac{1}{r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + 2\mu(\alpha+\beta)\frac{1}{r}\left(\frac{\partial\varphi}{\partial r}\right) \tag{D.17}$$

$$\sigma_z = \lambda\beta\nabla^2\varphi - 2\mu\alpha\nabla^2\left(\frac{\partial}{\partial z}\Psi_z\right) + 2\mu(\alpha+\beta)\frac{\partial^2\varphi}{\partial z^2}. \tag{D.18}$$

By replacing α with $-\frac{1}{\mu}$, β with $\frac{1}{\lambda+2\mu}$ and λ with $\frac{2\mu v}{1-2v}$, Eqs. (D.16)-(D.18) transform to

$$\sigma_r = 2\frac{\partial}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + \frac{1}{1-v}\left(v\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varphi \tag{D.19}$$

$$\sigma_\theta = \frac{2}{r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + \frac{1}{1-v}\left(v\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\varphi \tag{D.20}$$

$$\sigma_z = 2\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right) + \frac{1}{1-v}\left(v\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\varphi. \tag{D.21}$$

The same substitution into Eq. (D.12) (shear stress) results in,

$$\sigma_{rz} = \frac{\partial}{\partial r}\left(\nabla^2\Psi_z\right) + \frac{\partial}{\partial z}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r - \frac{1}{1-v}\frac{\partial^2\varphi}{\partial r\partial z}. \tag{D.22}$$

Eqs. (D.19)-(D.22) define the stress components in terms of stress functions.

APPENDIX E
STRESSES IN TERMS OF FOURIER BESSEL KERNELS

In this appendix stresses are formulated in terms of Fourier Bessel kernels and trigonometric functions.

Stress functions $\Psi_{r(r,z)}$ and $\Psi_{z(r,z)}$ are defined by $\int_{-L}^{L}\int_{a}^{b} X_{r(r',z')} \mathrm{G}_{r(r,r',z,z')} dr'dz'$ and $\int_{-L}^{L}\int_{a}^{b} X_{z(r',z')} \mathrm{G}_{z(r,r',z,z')} dr'dz'$ where $G_{r(r,r',z,z')}$ and $G_{z(r,r',z,z')}$ are radial and axial Green's functions, respectively, given by Eqs. (E.1) and (E.2).

$$G_{r(r,r',z,z')} = -\sum_{i=1}^{\infty}\left\{\frac{\pi^2}{2}\frac{\zeta_{1i}^2 J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)} r' K_1(\zeta_{1i}, r) K_1(\zeta_{1i}, r') \right. \tag{E.1}$$

$$\left. \left[\frac{1}{2L\zeta_{1i}^4} + L^3 \sum_{n=1}^{\infty} \frac{1}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2} \cos\frac{n\pi z'}{L} \cos\frac{n\pi z}{L}\right]\right\}$$

$$G_{z(r,r',z,z')} = -\frac{\pi^2}{2}\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{L^3}{\left(L^2\zeta_{0i}^2 + n^2\pi^2\right)^2}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)} \right. \tag{E.2}$$

$$\left. r' K_0(\zeta_{0i}, r') K_0(\zeta_{0i}, r) \sin\frac{n\pi z'}{L} \sin\frac{n\pi z}{L}\right]$$

From Eqs. (E.1) and (E.2) and by introducing new functions $\Gamma_{r(\zeta_{1i},n)}$ and $\Gamma_{z(\zeta_{0i},n)}$, the stress functions may be written as

$$\Psi_{r(r,z)} = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} \Gamma_{r(\zeta_{1i},n)} K_1(\zeta_{1i}, r) \cos\frac{n\pi z}{L} \tag{E.3}$$

$$\Psi_{z(r,z)} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \Gamma_{z(\zeta_{0i},n)} K_0(\zeta_{0i}, r) \sin\frac{n\pi z}{L} \tag{E.4}$$

where $\Gamma_{r(\zeta_{1i},n)}$ and $\Gamma_{z(\zeta_{0i},n)}$ are given by Eqs. (E.5) and (E.6).

$$\Gamma_{r(\zeta_{1i},n)} = -\left\{\frac{\pi^2}{2}\frac{\zeta_{1i}^2 J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)}\frac{L^3}{\left(L^2\zeta_{1i}^2 + n^2\pi^2\right)^2}\right. \tag{E.5}$$

$$\left.\int_{-L}^{L}\int_a^b r' X_{r(r',z')} K_{1(\zeta_{1i},r')} \cos\frac{n\pi z'}{L} dr' dz'\right\}$$

$$\Gamma_{r(\zeta_{1i},0)} = -\frac{\pi^2}{2}\frac{J_1^2(\zeta_{1i}a)}{J_1^2(\zeta_{1i}a) - J_1^2(\zeta_{1i}b)}\frac{1}{2L\zeta_{1i}^2}\int_{-L}^{L}\int_a^b r' X_{r(r',z')} K_{1(\zeta_{1i},r')} dr' dz'$$

$$\Gamma_{z(\zeta_{0i},n)} = -\frac{\pi^2}{2}\left\{\frac{L^3}{\left(L^2\zeta_{0i}^2 + n^2\pi^2\right)^2}\frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)}\right. \tag{E.6}$$

$$\left.\int_{-L}^{L}\int_a^b r' X_{z(r',z')} K_{0(\zeta_{0i},r')} \sin\frac{n\pi z'}{L} dr' dz'\right\}$$

The components of stress in terms of stress functions are given by Eqs. (E.7)-(E.10), where in these equations φ is given by Eq. (E.11).

$$\sigma_r = 2\frac{\partial}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + \frac{1}{1-\nu}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varphi \tag{E.7}$$

$$\sigma_\theta = \frac{2}{r}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r + \frac{1}{1-\nu}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\varphi \tag{E.8}$$

$$\sigma_z = 2\frac{\partial}{\partial z}\left(\nabla^2\Psi_z\right) + \frac{1}{1-\nu}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\varphi \tag{E.9}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left(\nabla^2\Psi_z\right) + \frac{\partial}{\partial z}\left(\nabla^2 - \frac{1}{r^2}\right)\Psi_r - \frac{1}{1-\nu}\frac{\partial^2\varphi}{\partial r\partial z} \tag{E.10}$$

$$\varphi = \frac{1}{r}\frac{\partial}{\partial r}(r\Psi_r) + \frac{\partial\Psi_z}{\partial z} \tag{E.11}$$

The substitution of Eqs. (E.3) and (E.4) into Eqs. (E.7)-(E.10) gives

$$\sigma_r = 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{\partial}{\partial r}\left(\nabla^2-\frac{1}{r^2}\right)\left[K_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right] \tag{E.12}$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\frac{1}{r}\frac{\partial}{\partial r}\left[rK_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\frac{\partial}{\partial z}\left[K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right]$$

$$\sigma_\theta = 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{1}{r}\left(\nabla^2-\frac{1}{r^2}\right)\left[K_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right] \tag{E.13}$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left(\nu\nabla^2-\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{1}{r}\frac{\partial}{\partial r}\left[rK_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(\nu\nabla^2-\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial}{\partial z}\left[K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right]$$

$$\sigma_z = 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{\partial}{\partial z}\nabla^2\left[K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right] \tag{E.14}$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left(\nu\nabla^2-\frac{\partial^2}{\partial z^2}\right)\frac{1}{r}\frac{\partial}{\partial r}\left[rK_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\frac{\partial}{\partial z}\left[K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right]$$

$$\sigma_{rz} = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{\partial}{\partial r}\nabla^2\left[K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right] \tag{E.15}$$

$$+\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{\partial}{\partial z}\left(\nabla^2-\frac{1}{r^2}\right)\left[K_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$-\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{\partial^2}{\partial r\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\left[rK_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$-\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{\partial^2}{\partial r\partial z}\frac{\partial}{\partial z}\left[K_{0(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right].$$

The Fourier Bessel kernel $K_{1(\zeta_{1i},r)}$ is obtained from Bessel functions of order one. Hence, it satisfies Bessel's equation which translates to

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\right]K_{1(\zeta_{1i},r)}=-\zeta_{1i}^2K_{1(\zeta_{1i},r)}. \tag{E.16}$$

Also from basic identities for Bessel functions, $J_n'(\zeta_{1i}r) = J_{n-1}(\zeta_{1i}r) - \frac{n}{\zeta_{1i}r}J_n(\zeta_{1i}r)$ and $Y_n'(\zeta_{1i}r) = Y_{n-1}(\zeta_{1i}r) - \frac{n}{\zeta_{1i}r}Y_n(\zeta_{1i}r)$, we obtain

$$\frac{\partial}{\partial r}K_{1(\zeta_{1i},r)}=\zeta_{1i}\left[J_{0(\zeta_{1i}r)}Y_{1(\zeta_{1i}b)}-J_{0(\zeta_{1i}b)}Y_{1(\zeta_{1i}r)}\right]-\frac{1}{r}K_{1(\zeta_{1i},r)}. \tag{E.17}$$

Likewise for $K_{0(\zeta_{0i},r)}$ we may write

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]K_{0(\zeta_{0i},r)}=-\zeta_{0i}^2K_{0(\zeta_{0i},r)} \tag{E.18}$$

$$\frac{\partial}{\partial r}K_{0(\zeta_{0i},r)}=-\zeta_{0i}\left[J_{1(\zeta_{0i}r)}Y_{0(\zeta_{0i}b)}-J_{0(\zeta_{0i}b)}Y_{1(\zeta_{0i}r)}\right]. \tag{E.19}$$

Using results of Eqs. (E.16)-(E.19) and introducing new kernels $K_{01(\zeta_{1i},r)}$ and $K_{10(\zeta_{0i},r)}$, Eqs. (E.12)-(E.15) are reduced to

$$\begin{aligned}
\sigma_r = {} & 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left[\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)\left(\zeta_{1i}K_{01(\zeta_{1i},r)}-\frac{1}{r}K_{1(\zeta_{1i},r)}\right)\cos\frac{n\pi z}{L}\right] \\
& +\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left[\nu\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)\zeta_{1i}K_{01(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right] \\
& +\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left[\zeta_{1i}^2\left(\zeta_{1i}K_{01(\zeta_{1i},r)}-\frac{1}{r}K_{1(\zeta_{1i},r)}\right)\cos\frac{n\pi z}{L}\right] \\
& +\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(\frac{n\pi}{L}\right)\left[\nu\left(-\zeta_{0i}^2-\frac{n^2\pi^2}{L^2}\right)K_{0(\zeta_{0i},r)}\cos\frac{n\pi z}{L}\right] \\
& +\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(\frac{n\pi}{L}\right)\left[\zeta_{0i}\left(\zeta_{0i}K_{0(\zeta_{0i},r)}-\frac{1}{r}K_{10(\zeta_{0i},r)}\right)\cos\frac{n\pi z}{L}\right]
\end{aligned} \tag{E.20}$$

$$\sigma_\theta = 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{1}{r}\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)\left[K_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right] \tag{E.21}$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\nu\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)\left[\zeta_{1i}K_{01(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{1}{r}\left[\zeta_{1i}^2K_{1(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\nu\left(-\zeta_{0i}^2-\frac{n^2\pi^2}{L^2}\right)\left[K_{0(\zeta_{0i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\frac{1}{r}\left[\zeta_{0i}K_{10(\zeta_{0i},r)}\cos\frac{n\pi z}{L}\right]$$

$$\sigma_z = 2\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\left(-\zeta_{0i}^2-\frac{n^2\pi^2}{L^2}\right)\left[K_{0(\zeta_{0i},r)}\cos\frac{n\pi z}{L}\right] \tag{E.22}$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\nu\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)\left[\zeta_{1i}K_{01(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\left(\frac{n\pi}{L}\right)^2\left[\zeta_{1i}K_{01(\zeta_{1i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\nu\left(-\zeta_{0i}^2-\frac{n^2\pi^2}{L^2}\right)\left[K_{0(\zeta_{0i},r)}\cos\frac{n\pi z}{L}\right]$$

$$+\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\zeta_{0i}\left[\left(\zeta_{0i}K_{0(\zeta_{0i},r)}-\frac{1}{r}K_{10(\zeta_{0i},r)}\right)\cos\frac{n\pi z}{L}\right]$$

$$\sigma_{rz} = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(-\zeta_{0i}^2-\frac{n^2\pi^2}{L^2}\right)\left[-\zeta_{0i}K_{10(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right] \tag{E.23}$$

$$+\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{n\pi}{L}\left(-\zeta_{1i}^2-\frac{n^2\pi^2}{L^2}\right)\left[K_{1(\zeta_{1i},r)}\sin\frac{n\pi z}{L}\right]$$

$$-\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{r(\zeta_{1i},n)}\frac{n\pi}{L}\left[-\zeta_{1i}^2K_{1(\zeta_{1i},r)}\sin\frac{n\pi z}{L}\right]$$

$$-\frac{1}{1-\nu}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\Gamma_{z(\zeta_{0i},n)}\left(\frac{n\pi}{L}\right)^2\left[\zeta_{0i}K_{10(\zeta_{0i},r)}\sin\frac{n\pi z}{L}\right]$$

where $K_{01(\zeta_{1i},r)}$ and $K_{10(\zeta_{0i},r)}$ are given by Eq. (E.24) and (E.25).

$$K_{01}(\zeta_{1i},r) = \left[J_0(\zeta_{1i}r)Y_1(\zeta_{1i}b) - J_1(\zeta_{1i}b)Y_0(\zeta_{1i}r)\right] \tag{E.24}$$

$$= \frac{1}{r\zeta_{1i}}K_1(\zeta_{1i},r) + \frac{1}{\zeta_{1i}}\frac{\partial}{\partial r}K_1(\zeta_{1i},r)$$

$$K_{10}(\zeta_{0i},r) = \left[J_1(\zeta_{0i}r)Y_0(\zeta_{0i}b) - J_0(\zeta_{0i}b)Y_1(\zeta_{0i}r)\right] = -\frac{1}{\zeta_{0i}}\frac{\partial}{\partial r}K_0(\zeta_{0i},r) \tag{E.25}$$

Note that to derive Eqs. (E-20)-(E.23), the following identities were used.

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rK_1(\zeta_{1i},r)\right] = \zeta_{1i}K_{01}(\zeta_{1i},r)$$

$$\nabla^2 K_{01}(\zeta_{1i},r) = -\zeta_{1i}^2 K_{01}(\zeta_{1i},r)$$

$$\frac{\partial}{\partial r}K_{01}(\zeta_{1i},r) = -\zeta_{1i}K_1(\zeta_{1i},r)$$

$$\frac{\partial^2}{\partial r^2}K_{01}(\zeta_{1i},r) = -\zeta_{1i}\left[\zeta_{1i}K_{01}(\zeta_{1i},r) - \frac{1}{r}K_1(\zeta_{1i},r)\right]$$

$$\frac{\partial}{\partial r}K_0(\zeta_{0i},r) = -\zeta_{0i}K_{10}(\zeta_{0i},r)$$

$$\frac{\partial^2}{\partial r^2}K_0(\zeta_{0i},r) = -\zeta_{0i}^2 K_0(\zeta_{0i},r) + \frac{1}{r}\zeta_{0i}K_{10}(\zeta_{0i},r)$$

Eqs. (E-20)-(E.23) are simplified further by combining the summations and canceling the common terms.

$$\sigma_r = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{r}\frac{1}{1-\nu}\left\{-\Gamma_z(\zeta_{0i},n)\frac{n\pi}{L}\left[\nu\frac{n^2\pi^2}{L^2} - \zeta_{0i}^2(1-\nu)\right]rK_0(\zeta_{0i},r)\right. \tag{E.26}$$

$$-\Gamma_z(\zeta_{0i},n)\zeta_{0i}\frac{n\pi}{L}K_{10}(\zeta_{0i},r)$$

$$-\Gamma_r(\zeta_{1i},n)\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(2-\nu) + \zeta_{1i}^2(1-\nu)\right]rK_{01}(\zeta_{1i},r)$$

$$\left.+\Gamma_r(\zeta_{1i},n)\left[2\frac{n^2\pi^2}{L^2}(1-\nu) + \zeta_{1i}^2(1-2\nu)\right]K_1(\zeta_{1i},r)\right\}\cos\frac{n\pi z}{L}$$

$$\sigma_\theta = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{r}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\nu\frac{n\pi}{L}\left(\frac{n^2\pi^2}{L^2}+\zeta_{0i}^2\right)rK_{0(\zeta_{0i},r)}\right.$$
$$+\Gamma_{z(\zeta_{0i},n)}\zeta_{0i}\frac{n\pi}{L}K_{10(\zeta_{0i},r)}$$
$$-\Gamma_{r(\zeta_{1i},n)}\nu\zeta_{1i}\left(\frac{n^2\pi^2}{L^2}+\zeta_{1i}^2\right)rK_{01(\zeta_{1i},r)}$$
$$\left.-\Gamma_{r(\zeta_{1i},n)}\left[2\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{1i}^2(1-2\nu)\right]K_{1(\zeta_{1i},r)}\right\}\cos\frac{n\pi z}{L} \tag{E.27}$$

$$\sigma_z = \sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\frac{n\pi}{L}\left[\frac{n^2\pi^2}{L^2}(1-\nu)+\zeta_{0i}^2(2-\nu)\right]K_{0(\zeta_{0i},r)}\right.$$
$$\left.+\Gamma_{r(\zeta_{1i},n)}\zeta_{1i}\left[\frac{n^2\pi^2}{L^2}(1-\nu)-\nu\zeta_{1i}^2\right]K_{01(\zeta_{1i},r)}\right\}\cos\frac{n\pi z}{L} \tag{E.28}$$

$$\sigma_{rz} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{1-\nu}\left\{-\Gamma_{z(\zeta_{0i},n)}\zeta_{0i}\left[\nu\frac{n^2\pi^2}{L^2}-\zeta_{0i}^2(1-\nu)\right]K_{10(\zeta_{0i},r)}\right.$$
$$\left.+\frac{n\pi}{L}\Gamma_{r(\zeta_{1i},n)}\left[\frac{n^2\pi^2}{L^2}(1-\nu)-\nu\zeta_{1i}^2\right]K_{1(\zeta_{1i},r)}\right\}\sin\frac{n\pi z}{L} \tag{E.29}$$

Eqs. (E.26)-(E.29) describe the components of stress in terms of Fourier Bessel kernels and trigonometric functions.

APPENDIX F
EQUATIONS FOR THE ARBITRARY CONSTANTS OF THE COMPLEMENTARY SOLUTION

In this appendix radial boundary conditions for $\xi_{(r,z)}$ are applied and four linear equations for the arbitrary constants of $\xi_{(r,z)}$ are derived

Expression for $\xi_{(r,z)}$ is given by Eq. (F-1), where A_i, $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ are arbitrary constants.

$$\xi_{(r,z)} = \pi^2 \sum_{i=1}^{\infty} \frac{\zeta_{0i}^2 J_0^2(\zeta_{0i}a)}{J_0^2(\zeta_{0i}a) - J_0^2(\zeta_{0i}b)} A_i\left[\sinh(\zeta_{0i}z) + \omega(\zeta_{oi})z\cosh(\zeta_{0i}z)\right]K_0(\zeta_{0i}, r) \tag{F.1}$$

$$+\frac{1}{L}\sum_{n=1}^{\infty}\left[\hat{C}_n I_0(\tfrac{n\pi}{L}r) + \hat{D}_n K_0(\tfrac{n\pi}{L}r)\right.$$

$$\left. + \frac{L}{2n\pi}\hat{A}_n r I_1(\tfrac{n\pi}{L}r) + \frac{L}{2n\pi}\hat{B}_n r K_1(\tfrac{n\pi}{L}r)\right]\sin\frac{n\pi z}{L}$$

Radial boundary conditions for $\xi_{(r,z)}$ are given by Eqs. (F.2)-(F.5).

$$\frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\xi_{(r,z)}\Bigg|_{r=a} = -\sum_{n=0}^{\infty}\wp_1(n)\cos\frac{n\pi z}{L} \tag{F.2}$$

$$\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\xi_{(r,z)}\Bigg|_{r=a} = -\sum_{n=1}^{\infty}\wp_2(n)\sin\frac{n\pi z}{L} \tag{F.3}$$

$$\frac{1}{1-\nu}\frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\xi_{(r,z)}\Bigg|_{r=b} = -\sum_{n=0}^{\infty}\wp_3(n)\cos\frac{n\pi z}{L} \tag{F.4}$$

$$\frac{1}{1-\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\xi_{(r,z)}\bigg|_{r=b}=-\sum_{n=1}^{\infty}\wp_{4(n)}\sin\frac{n\pi z}{L} \tag{F.5}$$

where $\wp_{1(n)}$ through $\wp_{4(n)}$ are known functions.

From Eq. (F.1) and the definitions for the Fourier Bessel kernels, we may write the following.

$$\frac{\partial}{\partial z}\nabla^2\xi_{(r,z)}\bigg|_{r=b}=\frac{n\pi}{L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{0(\frac{n\pi}{L}b)}+\hat{B}_n K_{0(\frac{n\pi}{L}b)}\right]\cos\frac{n\pi z}{L} \tag{F.6}$$

$$\frac{\partial}{\partial r}\nabla^2\xi_{(r,z)}\bigg|_{r=a}=-\frac{\pi}{a}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2-J_{0(\zeta_{0i}b)}^2}C_i\sinh(\zeta_{0i}z) \tag{F.7}$$

$$+\frac{n\pi}{L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{1(\frac{n\pi}{L}a)}+\hat{B}_n K_{1(\frac{n\pi}{L}a)}\right]\sin\frac{n\pi z}{L}$$

$$\frac{\partial}{\partial r}\nabla^2\xi_{(r,z)}\bigg|_{r=b}=-\frac{\pi}{b}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2-J_{0(\zeta_{0i}b)}^2}C_i\sinh(\zeta_{0i}z) \tag{F.8}$$

$$+\frac{n\pi}{L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{1(\frac{n\pi}{L}b)}+\hat{B}_n K_{1(\frac{n\pi}{L}b)}\right]\sin\frac{n\pi z}{L}$$

$$\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\xi_{(r,z)}\bigg|_{r=a}=-\frac{\pi}{a}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^4 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2-J_{0(\zeta_{0i}b)}^2}\left[A_i\sinh(\zeta_{0i}z)\right. \tag{F.9}$$

$$\left.+C_i z\cosh(\zeta_{0i}z)\right]$$

$$-\frac{\pi}{a}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2-J_{0(\zeta_{0i}b)}^2}C_i\sinh(\zeta_{0i}z)$$

$$-\frac{n^3\pi^3}{L^4}\sum_{n=1}^{\infty}\left[\hat{C}_n I_{1(\frac{n\pi}{L}a)}+\hat{D}_n K_{1(\frac{n\pi}{L}a)}\right.$$

$$\left.+\frac{L}{2n\pi}\hat{A}_n a I_{0(\frac{n\pi}{L}a)}+\frac{L}{2n\pi}\hat{B}_n a K_{0(\frac{n\pi}{L}a)}\right]\sin\frac{n\pi z}{L}$$

$$\left.\frac{\partial}{\partial z}\frac{\partial^2}{\partial r^2}\xi_{(r,z)}\right|_{r=a} = \frac{\pi}{a^2}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)} J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\left[\zeta_{0i}A_i\cosh(\zeta_{0i}z)\right.$$ (F.10)

$$\left.+C_i\left(\zeta_{0i}z\sinh(\zeta_{0i}z)+\cosh(\zeta_{0i}z)\right)\right]$$

$$+\frac{n^3\pi^3}{L^4}\sum_{n=1}^{\infty}\left[\hat{C}_n I_{0(\frac{n\pi}{L}a)} + \hat{D}_n K_{0(\frac{n\pi}{L}a)}\right.$$

$$\left.+\frac{L}{2n\pi}\hat{A}_n a I_{1(\frac{n\pi}{L}a)} + \frac{L}{2n\pi}\hat{B}_n a K_{1(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L}$$

$$-\frac{n^2\pi^2}{L^3}\sum_{n=1}^{\infty}\frac{1}{a}\left[\hat{C}_n I_{1(\frac{n\pi}{L}a)} + \hat{D}_n K_{1(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L}$$

$$+\frac{n\pi}{2L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{0(\frac{n\pi}{L}a)} + \hat{B}_n K_{0(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L}$$

$$\left.\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\xi_{(r,z)}\right|_{r=b} = -\frac{\pi}{b}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^4 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\left[A_i\sinh(\zeta_{0i}z)+C_i z\cosh(\zeta_{0i}z)\right]$$ (F.11)

$$-\frac{\pi}{b}\sum_{i=1}^{\infty}\frac{4\zeta_{0i}^3 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}C_i\sinh(\zeta_{0i}z)$$

$$-\frac{n^3\pi^3}{L^4}\sum_{n=1}^{\infty}\left[\hat{C}_n I_{1(\frac{n\pi}{L}b)} + \hat{D}_n K_{1(\frac{n\pi}{L}b)}\right.$$

$$\left.+\frac{L}{2n\pi}\hat{A}_n b I_{0(\frac{n\pi}{L}b)} + \frac{L}{2n\pi}\hat{B}_n b K_{0(\frac{n\pi}{L}b)}\right]\sin\frac{n\pi z}{L}$$

$$\left.\frac{\partial}{\partial z}\frac{\partial^2}{\partial r^2}\xi_{(r,z)}\right|_{r=b} = \frac{\pi}{b^2}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\left[\zeta_{0i}A_i\cosh(\zeta_{0i}z)\right.$$ (F.12)

$$\left.+C_i\left(\zeta_{0i}z\sinh(\zeta_{0i}z)+\cosh(\zeta_{0i}z)\right)\right]$$

$$+\frac{n^3\pi^3}{L^4}\sum_{n=1}^{\infty}\left[\hat{C}_n I_{0(\frac{n\pi}{L}b)} + \hat{D}_n K_{0(\frac{n\pi}{L}b)}\right.$$

$$\left.+\frac{L}{2n\pi}\hat{A}_n b I_{1(\frac{n\pi}{L}b)} + \frac{L}{2n\pi}\hat{B}_n b K_{1(\frac{n\pi}{L}b)}\right]\cos\frac{n\pi z}{L}$$

$$-\frac{n^2\pi^2}{L^3}\sum_{n=1}^{\infty}\frac{1}{b}\left[\hat{C}_n I_{1(\frac{n\pi}{L}b)} + \hat{D}_n K_{1(\frac{n\pi}{L}b)}\right]\cos\frac{n\pi z}{L}$$

$$+\frac{n\pi}{2L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{0(\frac{n\pi}{L}b)} + \hat{B}_n K_{0(\frac{n\pi}{L}b)}\right]\cos\frac{n\pi z}{L}$$

$$\left.\frac{\partial}{\partial z}\nabla^2\xi_{(r,z)}\right|_{r=a} = \frac{n\pi}{L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{0(\frac{n\pi}{L}a)} + \hat{B}_n K_{0(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L} \tag{F.13}$$

The substitution of Eqs. (F.13) and (F.10) into radial stress boundary condition at $r = a$, yields

$$\begin{aligned}
&\frac{1}{1-\nu}\left\{\nu\frac{n\pi}{L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{0(\frac{n\pi}{L}a)} + \hat{B}_n K_{0(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L}\right. \\
&\quad - \frac{n^3\pi^3}{L^4}\sum_{n=1}^{\infty}\left[\hat{C}_n I_{0(\frac{n\pi}{L}a)} + \hat{D}_n K_{0(\frac{n\pi}{L}a)} + \frac{L}{2n\pi}\hat{A}_n a I_{1(\frac{n\pi}{L}a)}\right. \\
&\quad\quad \left. + \frac{L}{2n\pi}\hat{B}_n a K_{1(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L} \\
&\quad + \frac{n^2\pi^2}{L^3}\sum_{n=1}^{\infty}\frac{1}{a}\left[\hat{C}_n I_{1(\frac{n\pi}{L}a)} + \hat{D}_n K_{1(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L} \\
&\quad - \frac{n\pi}{2L^2}\sum_{n=1}^{\infty}\left[\hat{A}_n I_{0(\frac{n\pi}{L}a)} + \hat{B}_n K_{0(\frac{n\pi}{L}a)}\right]\cos\frac{n\pi z}{L} \\
&\quad - \frac{\pi}{a^2}\sum_{i=1}^{\infty}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)} J_{0(\zeta_{0i}b)}}{J^2_{0(\zeta_{0i}a)} - J^2_{0(\zeta_{0i}b)}} A_i\left[\zeta_{0i}\cosh(\zeta_{0i}z)\right. \\
&\quad\quad \left.\left. + \omega_{(\zeta_{oi})}\left(\zeta_{0i}z\sinh(\zeta_{0i}z) + \cosh(\zeta_{0i}z)\right)\right]\right\} = -\sum_{n=1}^{\infty}\wp_{1(n)}\cos\frac{n\pi z}{L}.
\end{aligned} \tag{F.14}$$

Using the orthogonality property of cosine and introducing new functions $\Lambda_{1(\zeta_{ai},n)}$, $\bar{\lambda}_{11(n)}$, $\bar{\lambda}_{12(n)}$, $\bar{\lambda}_{13(n)}$ and $\bar{\lambda}_{14(n)}$, Eq. (F.14) reduces to Eq. (F.15).

$$\bar{\lambda}_{11(n)}\hat{A}_n + \bar{\lambda}_{12(n)}\hat{B}_n + \bar{\lambda}_{13(n)}\hat{C}_n + \bar{\lambda}_{14(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{1(\zeta_{0i},n)}A_i = \wp_{1(n)} \tag{F.15}$$

The definitions for $\bar{\lambda}_{11(n)}$, $\bar{\lambda}_{12(n)}$, $\bar{\lambda}_{13(n)}$, $\bar{\lambda}_{14(n)}$ and $\Lambda_{1(\zeta_{ai},n)}$ are given by Eqs. (F.16) through (F.20), respectively.

$$\bar{\lambda}_{11(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu - \frac{1}{2}\right)\frac{n\pi}{L^2}I_{0(\frac{n\pi}{L}a)} - \frac{n^2\pi^2}{2L^3}aI_{1(\frac{n\pi}{L}a)}\right] \tag{F.16}$$

$$\lambda_{12(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu - \frac{1}{2}\right)\frac{n\pi}{L^2}K_{0(\frac{n\pi}{L}a)} - \frac{n^2\pi^2}{2L^3}aK_{1(\frac{n\pi}{L}a)}\right] \tag{F.17}$$

$$\lambda_{13(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}I_{0(\frac{n\pi}{L}a)} + \frac{n^2\pi^2}{L^3}\frac{1}{a}I_{1(\frac{n\pi}{L}a)}\right] \tag{F.18}$$

$$\lambda_{14(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}K_{0(\frac{n\pi}{L}a)} + \frac{n^2\pi^2}{L^3}\frac{1}{a}K_{1(\frac{n\pi}{L}a)}\right] \tag{F.19}$$

$$\Lambda_{1(\zeta_{oi},n)} = \frac{1}{(1-\nu)}\frac{\pi}{a^2}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)}J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\Big[\zeta_{0i}\Omega_{2(\zeta_{oi},n)} + \omega_{(\zeta_{oi})}\left(\zeta_{0i}\Omega_{1(\zeta_{oi},n)} + \Omega_{2(\zeta_{oi},n)}\right)\Big] \tag{F.20}$$

Where in Eq. (F.20), $\Omega_{1(\zeta_{0i},n)}$ and $\Omega_{2(\zeta_{0i},n)}$ are given by Eqs. (F.21) and (F.22).

$$\int_{-L}^{L} z\cos\frac{n\pi z}{L}\sinh(\zeta_{0i}z)dz = \frac{2L^2(-1)^n}{\left(\zeta_{0i}^2L^2 + n^2\pi^2\right)^2}\Big[-\left(\zeta_{0i}^2L^2 - n^2\pi^2\right)\sinh(\zeta_{0i}L) + \left(\zeta_{0i}^2L^2 + n^2\pi^2\right)\zeta_{0i}L\cosh(\zeta_{0i}L)\Big] = \Omega_{1(\zeta_{0i},n)} \tag{F.21}$$

$$\int_{-L}^{L} \cos\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz = \frac{2\zeta_{0i}L^2(-1)^n}{\left(\zeta_{0i}^2L^2 + n^2\pi^2\right)}\sinh(\zeta_{0i}L) = \Omega_{2(\zeta_{0i},n)} \tag{F.22}$$

The substitution of Eqs. (F.6) and (F.12) into the radial stress boundary condition at $r = b$, yields

$$\lambda_{21(n)}\hat{A}_n + \lambda_{22(n)}\hat{B}_n + \lambda_{23(n)}\hat{C}_n + \lambda_{24(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{2(\zeta_{0i},n)}A_i = \wp_{2(n)} \tag{F.23}$$

where $\lambda_{21(n)}$, $\lambda_{22(n)}$, $\lambda_{23(n)}$, $\lambda_{24(n)}$ and $\Lambda_{2(\zeta_{oi},n)}$ are given by Eqs. (F.24) through (F.28), respectively.

$$\lambda_{21(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu - \frac{1}{2}\right)\frac{n\pi}{L^2}I_{0(\frac{n\pi}{L}b)} - \frac{n^2\pi^2}{2L^3}bI_{1(\frac{n\pi}{L}b)}\right] \tag{F.24}$$

$$\lambda_{22(n)} = \frac{-1}{(1-\nu)}\left[\left(\nu - \frac{1}{2}\right)\frac{n\pi}{L^2}K_{0(\frac{n\pi}{L}b)} - \frac{n^2\pi^2}{2L^3}bK_{1(\frac{n\pi}{L}b)}\right] \tag{F.25}$$

$$\lambda_{23(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}I_{0(\frac{n\pi}{L}b)} + \frac{n^2\pi^2}{L^3}\frac{1}{b}I_{1(\frac{n\pi}{L}b)}\right] \tag{F.26}$$

$$\lambda_{24(n)} = \frac{-1}{(1-\nu)}\left[-\frac{n^3\pi^3}{L^4}K_{0(\frac{n\pi}{L}b)} + \frac{n^2\pi^2}{L^3}\frac{1}{b}K_{1(\frac{n\pi}{L}b)}\right] \tag{F.27}$$

$$\Lambda_{2(\zeta_{oi},n)} = \frac{1}{(1-\nu)}\frac{\pi}{b^2}\frac{2\zeta_{0i}^2 J_{0(\zeta_{0i}a)}^2}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\Big[\zeta_{0i}\Omega_{2(\zeta_{oi},n)} + \omega_{(\zeta_{oi})}\left(\zeta_{0i}\Omega_{1(\zeta_{oi},n)} + \Omega_{2(\zeta_{oi},n)}\right)\Big] \tag{F.28}$$

The substitution of Eqs. (F.7) and (F.9) into shear stress boundary condition at $r = a$, yields

$$\lambda_{31(n)}\hat{A}_n + \lambda_{32(n)}\hat{B}_n + \lambda_{33(n)}\hat{C}_n + \lambda_{34(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{3(\zeta_{0i},n)}A_i = \wp_{3(n)} \tag{F.29}$$

where $\lambda_{31(n)}$, $\lambda_{32(n)}$, $\lambda_{33(n)}$, $\lambda_{34(n)}$ and $\Lambda_{3(\zeta_{oi},n)}$ are expressed by Eq. (F.30)-(F.34).

$$\lambda_{31(n)} = -\left[\frac{n\pi}{L^2}I_{1(\frac{n\pi}{L}a)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}aI_{0(\frac{n\pi}{L}a)}\right] \tag{F.30}$$

$$\lambda_{32(n)} = -\left[\frac{n\pi}{L^2}K_{1(\frac{n\pi}{L}a)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}aK_{0(\frac{n\pi}{L}a)}\right] \tag{F.31}$$

$$\lambda_{33(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}I_{1(\frac{n\pi}{L}a)}\right] \tag{F.32}$$

$$\lambda_{34(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}K_{1(\frac{n\pi}{L}a)}\right] \tag{F.33}$$

$$\Lambda_{3(\zeta_{oi},n)} = \frac{1}{1-\nu}\frac{\pi}{a}\frac{2\zeta_{0i}^3 J_{0(\zeta_{0i}a)} J_{0(\zeta_{0i}b)}}{J_{0(\zeta_{0i}a)}^2 - J_{0(\zeta_{0i}b)}^2}\Big[2\nu\omega_{(\zeta_{oi})}\Omega_{4(\zeta_{oi},n)} - \zeta_{0i}\left(\Omega_{4(\zeta_{oi},n)} + \omega_{(\zeta_{oi})}\Omega_{3(\zeta_{oi},n)}\right)\Big] \tag{F.34}$$

The definitions for $\Omega_{3(\zeta_{0i},n)}$ and $\Omega_{4(\zeta_{0i},n)}$ are given by Eqs. (F.35) and (F.36).

$$\int_{-L}^{L} z\sin\frac{n\pi z}{L}\cosh(\zeta_{0i}z)dz = \frac{2L^2(-1)^n}{\left(\zeta_{0i}^2L^2+n^2\pi^2\right)^2}\Big[2n\pi L\zeta_{0i}\sinh(\zeta_{0i}L) - \left(\zeta_{0i}^2L^2+n^2\pi^2\right)n\pi\cosh(\zeta_{0i}L)\Big] = \Omega_{3(\zeta_{0i},n)} \tag{F.35}$$

$$\int_{-L}^{L}\sin\frac{n\pi z}{L}\sinh(\zeta_{0i}z)dz = \frac{-2n\pi L(-1)^n}{\left(\zeta_{0i}^2L^2+n^2\pi^2\right)}\sinh(\zeta_{0i}L) = \Omega_{4(\zeta_{0i},n)} \tag{F.36}$$

The substitution of Eqs. (F.8) and (F.11) into the shear stress boundary condition at $r = b$, yields

$$\lambda_{41(n)}\hat{A}_n + \lambda_{42(n)}\hat{B}_n + \lambda_{43(n)}\hat{C}_n + \lambda_{44(n)}\hat{D}_n + \sum_{i=1}^{\infty}\Lambda_{4(\zeta_{0i},n)}A_i = \wp_{4(n)} \tag{F.37}$$

where $\lambda_{41(n)}$, $\lambda_{42(n)}$, $\lambda_{43(n)}$, $\lambda_{44(n)}$ and $\Lambda_{4(\zeta_{oi},n)}$ are defined by Eqs. (F.38)-(F.42).

$$\lambda_{41(n)} = -\left[\frac{n\pi}{L^2}I_{1(\frac{n\pi}{L}b)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}bI_{0(\frac{n\pi}{L}b)}\right] \tag{F.38}$$

$$\lambda_{42(n)} = -\left[\frac{n\pi}{L^2}K_{1(\frac{n\pi}{L}b)} + \frac{1}{(1-\nu)}\frac{n^2\pi^2}{2L^3}bK_{0(\frac{n\pi}{L}b)}\right] \tag{F.39}$$

$$\lambda_{43(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}I_{1(\frac{n\pi}{L}b)}\right] \tag{F.40}$$

$$\lambda_{44(n)} = -\left[\frac{n^3\pi^3}{L^4}\frac{1}{(1-\nu)}K_{1(\frac{n\pi}{L}b)}\right] \tag{F.41}$$

$$\Lambda_{4(\zeta_{oi},n)} = \frac{1}{1-\nu}\frac{\pi}{b}\frac{2\zeta_{0i}^{3}J_{0}^{2}(\zeta_{0i}a))}{J_{0}^{2}(\zeta_{0i}a) - J_{0}^{2}(\zeta_{0i}b)}\Big[2\nu\omega(\zeta_{oi})\Omega_{4(\zeta_{oi},n)} - \zeta_{0i}\big(\Omega_{4(\zeta_{oi},n)} + \omega(\zeta_{oi})\Omega_{3(\zeta_{oi},n)}\big)\Big] \tag{F.42}$$

Eqs. (F.15), (F.23), (F.29) and (F.37) represent four linear equations for the five arbitrary constants A_i, $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$.

LIST OF REFERENCES

1. Eshelby, J. D., "The Determination of the Elastic Field of an Ellipsoidal Inclusion, and Related Problems," Proceedings Royal Society, London, England, 1957.

2. Mori, T., and Tanaka, K., "Average Stress in Matrix and Average Elastic Energy of Materials with Misfitting Inclusions," Acta Metallurgica, 1973.

3. Hasegawa, H., Lee, V., and Mura, T., "Stress Fields Caused by a Circular Cylindrical Inclusion," American Society of Mechanical Engineers, Winter Annual Meeting Atlanta, GA, USA, 1991.

4. Huang, Y., Hu, K. X., Wei, X., and Chandra, A., "Generalized Self-Consistent Mechanics Method for Composite Materials with Multiphase Inclusions," Journal of the Mechanics and Physics of Solids , Vol. 42, No. 8, 1994.

5. Kuo, C. H., and Keer, L. M., "Three-Dimensional Analysis of Cracking in a Multilayered Composite," Journal of Applied Mechanics, Vol. 62, No. 2, 1995.

6. Hu, K. X., and Chandra, A., "Interactions Among General Systems of Cracks and Anticracks: An Integral Equation Approach," Journal of Applied Mechanics, Vol. 60, No. 4 , 1993.

7. Noda, N., and Matsuo, T., "Singular Integral Equation Method in Optimization of Stress-Relieving Hole: A New Approach Based on the Body Force Method," International Journal of Fracture, Vol. 70, No. 2, 1995.

8. Moon, F., "Magneto-Solid Mechanics," Wiley- Interscience, 1984.

9. Reed, R., and Clark, A., "Materials at Low Temperatures," American Society for Metals, 1983.

10. Wilson, M., "Superconducting Magnets," Clarendon Press Oxford, 1983.

11. Schelkunoff, S. A., "Electromagnetic Fields," Blaisdell Publishing Company, 1963.

12. Montgomery, D., "Solenoid Magnet Design," Wiley-Interscience, 1969.

13. Burkhard, H., "Magnetic and Stress-Field of Cylindrical Non-Uniform Current Density Superconducting Coils," Journal of Applied Physics, Vol. 6, 1975.

14. Thlen, K., and Schneider-Muntau, H., "Calculation of Non-Destructive Pulsed Field Magnets," Max-Planck Institute, Grenoble, Internal report.

15. Brechna, H., "Superconducting Magnet Systems," Springer Publishing Company, 1973.

16. Calcote, L., "Introduction to Continuum Mechanics," Van Nostrand Publishing Company, 1969.

17. Lontai, L. M., and Marston, P. G., "A 100 Kilogauss Quasi-Continuos Cryogenic Solenoid - Part I," Proceeding International Symposium on Magnet Technology, Stanford University, CA, USA, 1965.

18. Gray, W. H., and Ballou, J. K., "Electromechanical Stress Analysis of Transversely Isotropic Solenoids," Journal of Applied Physics, Vol. 48, No. 7, 1977.

19. Arp, V., "Stresses in Superconducting Solenoids," Journal of Applied Physics, Vol. 48, No. 5, 1977.

20. Mitchell, N., and Mszanowski, U., "Stress Analysis of Structurally Graded Long Solenoid Coils," IEEE Transactions on Magnets, Vol. 28, No. 1, 1992.

21. Markiewicz, W. D., Voleti, S. R., Chandra, N., and Murray, F. S., "Transverse Stress on Nb_3Sn Conductors in High Field NMR Magnets," IEEE Transactions on Applied Superconductivity, Vol. 3, No. 1, 1993.

22. Middleton, A. J., and Trowbridge, C. W., "Mechanical Stress in Large High Field Magnet Coils," Proceeding of the 2nd International Conference on Magnet Technology, Oxford, England, 1967.

23. Markiewicz, W. D., Vaghar, M. R., Dixon, I. R., and Garmestani, H., "Generalized Plane Strain Analysis of Solenoid Magnets," IEEE Transactions on Magnets, Vol. 30, No. 4, 1994.

24. Bobrov, E. S., "Electrically Conducting Orthotropic Cylinder Shell in Axial and Radial Magnetic Field," The Mechanical Behavior of Electromagnetic Solid Continua, G. A. Maugin, Ed., N-Holland: Elsevier Science Publisher, 1984.

25. Cox, A., Garmestani, H., Markiewicz, W. D., and Dixon, I. R., "Power Series Stress Analysis of Solenoid Magnets," International Conference on Magnet Technology, Finland, 1995.

26. Hasegawa, H., "Axisymmetric Body Force Problems of an Elastic Half Space," Japan Society of Mechanical Engineers, Vol. 19, No. 137, 1976.

27. Hasegawa, H., Lee, V., and Mura, T., "Green's Functions for Axisymmetric Problems of Dissimilar Elastic Solids," American Society of Mechanical Engineers, Summer Mechanics and Materials Meeting, Tempe, AZ, USA, 1992.

28. Tranter, C. J., "Integral Transforms in Mathematical Physics," Butler and Tanner Ltd., 1971.

29. Sneddon, I. N., "Fourier Transforms," McGraw-Hill, 1951.

30. Hibbitt, Karlsson, and Sorenson, Inc., ABAQUS, Version 5.2, Copyright 1992.

31. Rama, S. C., Voleti, S. R., and Markiewicz, W. D., " Integration of User Subroutines with Finite Element Packages Available at The NHMFL General Requirements," Internal Report # 787, National High Magnetic Field Laboratory, Tallahassee, FL, 1993.

32. Chadwick, P., "Continuum Mechanics," Wiley- Interscience, 1976.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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